

# Direct initial-algebra semantics for (multimodal) dependent type theories

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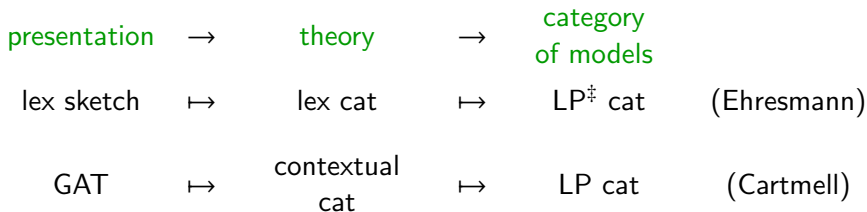
SYNTAX AND SEMANTICS OF TYPE THEORIES, Ljubljana, June 2026

# Goal

Specifying dependent type theories, in the spirit of **initial-algebra semantics** (Goguen, 1974).

- Define **category of “models”**.
- Desired type theory **implicitly** defined as **initial object** therein...
- ... up to isomorphism!

# The usual pattern

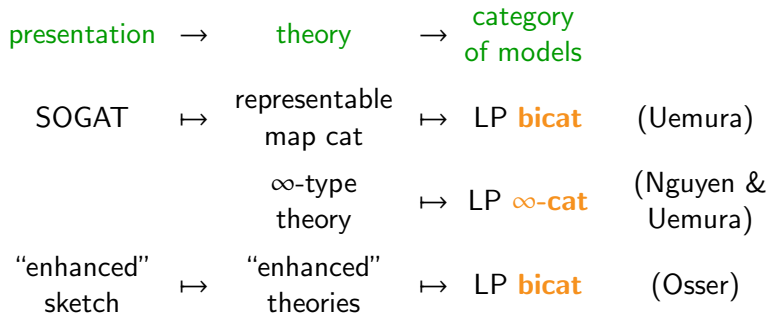


Not so convenient for specifying dependent type theories.

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<sup>‡</sup>LP = locally presentable

# The usual pattern, second generation



- More convenient for dependent type theories!
- Higher-dimensional.

# This work

We propose:

## Direct initial-algebra semantics

- Work directly in **RLPCAT**.
  - No presentations, no theories.
- 
- 1-categories of models.
  - Constructions similar to Nguyen and Uemura's.
  - In **RLPCAT**: LP categories, right adjoints.
  - More accessible: old-school category theory.
  - More precise (up to iso rather than  $\infty$ -equivalence).
  - Counterpart:
    - Isofibration condition in pullbacks.
    - Morphisms preserve operations strictly.

# Plan

- ① Intro
- ② By example
- ③ Local presentability
- ④ SOGATs
- ⑤ Conclusion
- ⑥ Bonus

# Example 0: simple operation

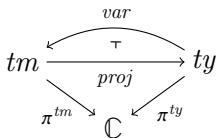
Pullback along suitable “refinement” functors.

$$\begin{array}{ccc}
 \bullet & \longrightarrow & \mathbf{Set}^2 \\
 \downarrow \lrcorner & & \downarrow \text{refinement} \\
 \mathcal{A} & \longrightarrow & \mathbf{Set}^2 \\
 A \vdash \longrightarrow & & A^2
 \end{array}
 \qquad
 \begin{array}{ccc}
 \bullet & \xrightarrow{p} & \bullet \\
 \downarrow & & \\
 \bullet & & \\
 & & \bullet \\
 & & A
 \end{array}$$

Object of the pullback:  $A$  equipped with  $A^2 \rightarrow A$ .

# Example I: natural models

## Natural model (Awodey, 2018)



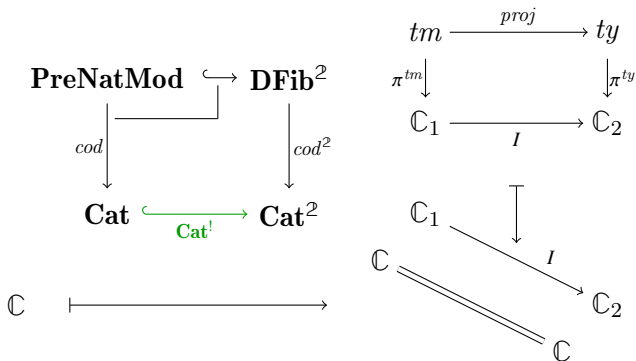
where  $\pi^{ty}$  and  $\pi^{tm}$  are discrete fibrations.

(*var* not necessarily over  $\mathbb{C}$ .)

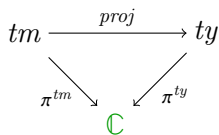
## Context extension *ext*

$$\begin{array}{ccccc}
 ty & \xrightarrow{\text{var}} & tm & \xrightarrow{\pi^{tm}} & \mathbb{C} \\
 \Gamma \vdash A & \mapsto & \Gamma, A \vdash \top : A & \mapsto & \Gamma, A.
 \end{array}$$

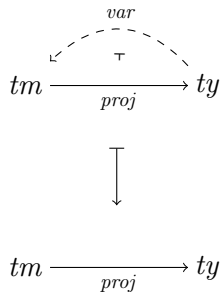
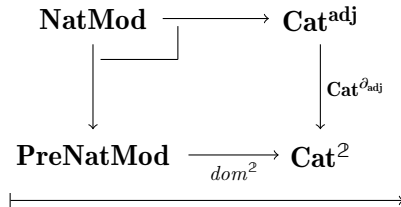
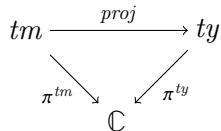
# Example I: natural models



Objects: **pre-natural models** = natural models without the right adjoint.



# Example I: natural models



## Example II: dependent products

- First recall standard definition as **extensional type constructor** (Coraglia and Di Liberti).
- Then reconstruct in **RLPCAT**.

Defined by

$$\begin{array}{ccc}
 \mathfrak{P}(tm) & \overset{\lambda}{\dashrightarrow} & tm \\
 \mathfrak{P}(proj) \downarrow & \dashrightarrow & \downarrow proj \\
 \mathfrak{P}(ty) & \overset{\Pi}{\dashrightarrow} & ty
 \end{array}$$

where  $\mathfrak{P} = \pi_!^{ty} \circ ext^*$ .

# Example II: dependent products

## Definition of $\mathfrak{P}$

$$\mathbf{DFib}_{\mathbb{C}} \xrightarrow{\text{ext}^*} \mathbf{DFib}_{ty} \xrightarrow{\pi_1^{ty}} \mathbf{DFib}_{\mathbb{C}}.$$

## Computing $\mathfrak{P}(ty)$

$$\begin{array}{ccccc}
 B & \longleftarrow & \vdash & \Gamma, A \vdash B & \\
 \downarrow \pi^{ty} & & & \downarrow & \swarrow \mathfrak{P}(\pi^{ty}) \\
 ty & \longleftarrow & \mathfrak{P}(ty) & & \\
 \downarrow \pi^{ty} & & \downarrow & & \searrow \pi^{ty} \\
 \mathbb{C} & \longleftarrow & \text{ext} & ty & \longrightarrow \mathbb{C} \\
 \Gamma, A & \longleftarrow & \vdash & \Gamma \vdash A & \longrightarrow \Gamma
 \end{array}$$

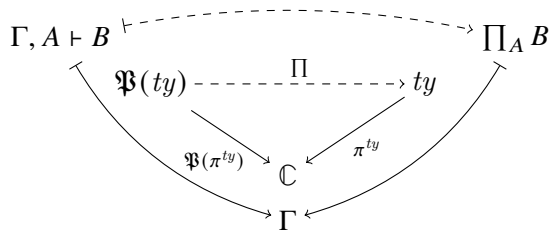
Element of  $\mathfrak{P}(ty)_{\Gamma}$ : a type  $\Gamma \vdash A$ , together with  $\Gamma, A \vdash B$ .

# Example II: dependent products

## Definition of $\mathfrak{P}$

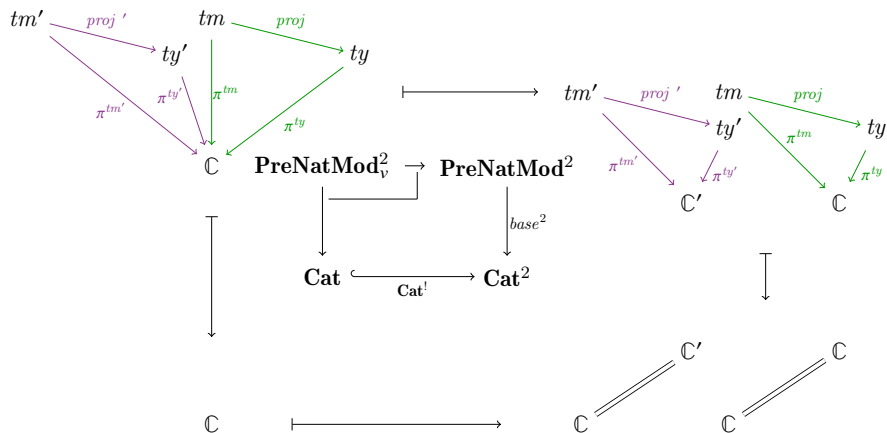
$$\mathbf{DFib}_{\mathbb{C}} \xrightarrow{\text{ext}^*} \mathbf{DFib}_{ty} \xrightarrow{\pi_!^{ty}} \mathbf{DFib}_{\mathbb{C}}.$$

## Type of $\Pi$



# Example II: dependent products

Reconstruction in **RLPCAT**, first step.



# Example II: dependent products

Reconstruction in **RLPCAT**, second step.

$$\begin{array}{ccc}
 \mathbf{PreNatMod}_v^{2_{\text{cart}}} & \xrightarrow{\quad} & \mathbf{PreNatMod}^{2_{\text{cart}}} \\
 \downarrow & \lrcorner & \downarrow \text{base}^{2_{\text{cart}}} \\
 \mathbf{Cat} & \xrightarrow{\text{Cat}^!} & \mathbf{Cat}^{2_{\text{cart}}}
 \end{array}$$

Similar...

Objects:

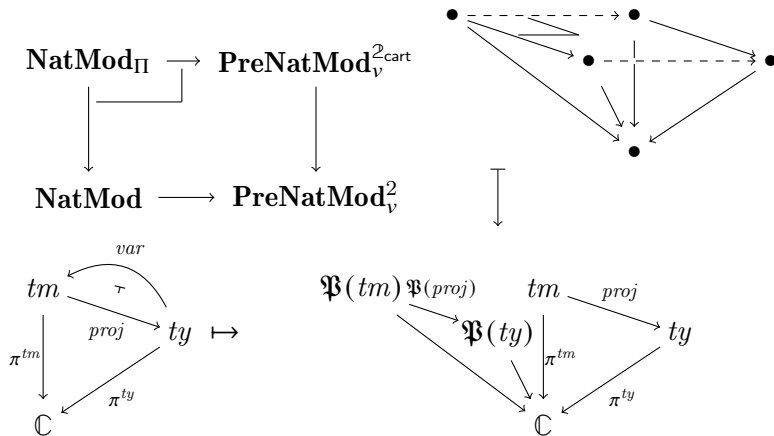
$$\begin{array}{ccccc}
 tm' & \xrightarrow{h} & tm & \xrightarrow{\text{proj}} & ty \\
 & \searrow & \swarrow & \searrow & \\
 & & ty' & \xrightarrow{k} & ty \\
 & \searrow & \downarrow \pi^{ty'} & \downarrow \pi^{tm} & \downarrow \pi^{ty} \\
 & & \mathbb{C} & & \mathbb{C}
 \end{array}$$

Additional arrows from  $tm'$  to  $\mathbb{C}$ :
 

- $\pi^{tm'}$  (purple arrow)
- $\pi^{ty'}$  (purple arrow)

# Example II: dependent products

Reconstruction in **RLPCAT**, last step.



# Plan

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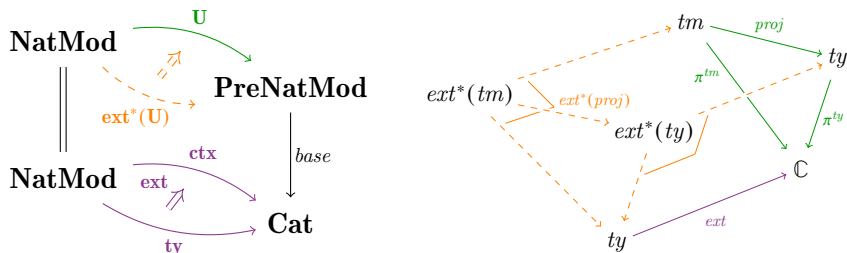
# Disclaimer

- Mere peek for lack of time.
- Plus it's work in progress.
- Focus on hardest part:  $\mathfrak{B} = \pi_1^{ty} \circ ext^*$ .

# $ext^*$ : cartesian lifting

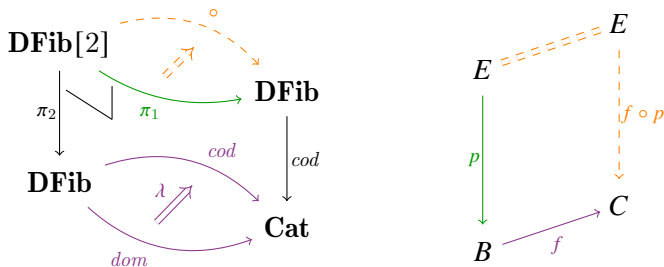
- Fibration  $base: \mathbf{PreNatMod} \rightarrow \mathbf{Cat}$ .
- Cartesian lifting by pullback.

Also a fibration in **RLPCAT** (Street, 1974):



# $\pi_!^{ty}$ : opcartesian lifting

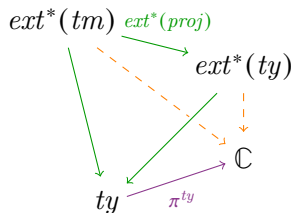
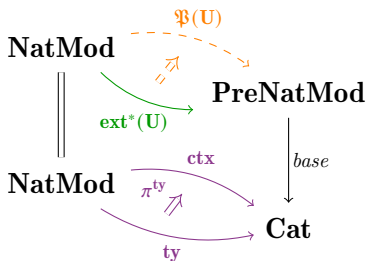
- Opfibration *base*: **PreNatMod**  $\rightarrow$  **Cat**.
- Problem: not an opfibration in **RLPCAT**.
- Current solution: show that “canonical” opcartesian lifting lives in **RLPCAT**.



(Discrete fibrations compose.)

# $\pi_!^{ty}$ : opcartesian lifting

Deduce desired opcartesian lifting.

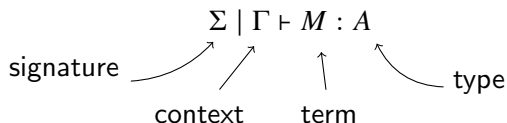


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# Brief reminder on SOGATs

- Based on a logical framework, say ULF (Uemura's LF).
- Two sorts:  $*$  and  $\square$ .
- Two-level “contexts”, as in

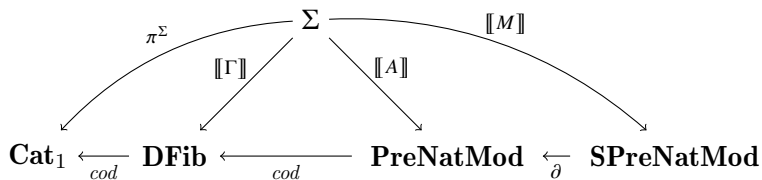


## Main entry point

Signature  $\leadsto$  category of models.

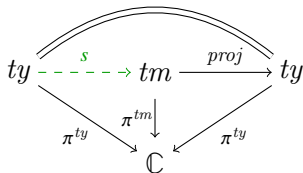
# Interpretation of judgements

Concretely,  $\Sigma | \Gamma \vdash M : A$  is interpreted as



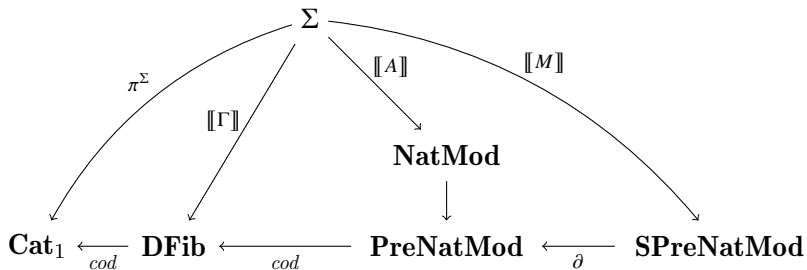
when  $A : \square$ , where

- $\mathbf{Cat}_1$ : categories with a terminal object;
- $\mathbf{SPreNatMod} :=$  pre-natural models **with a section**.



# Interpretation of judgements

Concretely,  $\Sigma | \Gamma \vdash M : A$  is interpreted as



when  $A : *$ .

# Signature extension (simplified)

Let  $s \in \{*, \square\}$ .

$$\frac{\Sigma | \Gamma \vdash}{\Sigma, \Gamma \Rightarrow s \vdash}$$

$$\frac{\Sigma | \Gamma \vdash A : \square}{\Sigma, \Gamma \Rightarrow A \vdash}$$

$$\frac{\Sigma | \Gamma \vdash}{\Sigma, \Gamma \Rightarrow s | \Gamma \vdash \top : s}$$

$$\frac{\Sigma | \Gamma \vdash A : \square}{\Sigma, \Gamma \Rightarrow A | \Gamma \vdash \top : A}$$

$$\begin{array}{ccc} \Sigma, \Gamma \Rightarrow s & \xrightarrow{\Sigma, \Gamma \Rightarrow s | \Gamma \vdash \top : s} & \mathbf{(Pre)NatMod} \\ \downarrow & \lrcorner & \downarrow \text{ty} \\ \Sigma & \xrightarrow{\Gamma} & \mathbf{DFib} \end{array}$$

$$\begin{array}{ccc} \Sigma, \Gamma \Rightarrow A & \xrightarrow{\Sigma, \Gamma \Rightarrow A | \Gamma \vdash \top : A} & \mathbf{SPreNatMod} \\ \downarrow & \lrcorner & \downarrow \partial \\ \Sigma & \xrightarrow{A} & \mathbf{PreNatMod} \end{array}$$

- The former adds “indexed stuff”.
- The latter adds “indexed structure”.

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# Conclusion

## Direct initial-algebra semantics

Construct categories of models directly from 2-categorical properties of **RLPCAT**.

No presentations, no theories.

- Use Nguyen and Uemura's techniques for the most part.
- Type constructors require more work.

Applications:

- SOGATs and
- (not here) multimodal type theory.

# Outlook

- Improve treatment of opcartesian lifting.
- Strengthen link with Uemura's semantics?

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# More on SOGATs

- For fixed signature  $\Sigma$ :
  - behaves like a normal type theory with dependent products, say  $ULF_{\Sigma}$ ,
  - with two sorts  $*$  and  $\square$ ,
  - $(*, \square, \square)$ -dependent products.

$$\frac{\Sigma|\Gamma \vdash A : * \quad \Sigma|\Gamma, A \vdash B : \square}{\Sigma|\Gamma \vdash \prod_A B : \square}$$

- Signature extension rules:

$$\frac{\Sigma|\Gamma \vdash}{\Sigma, \Gamma \Rightarrow * \vdash}$$

$$\frac{\Sigma|\Gamma \vdash}{\Sigma, \Gamma \Rightarrow \square \vdash}$$

$$\frac{\Sigma|\Gamma \vdash A : \square}{\Sigma, \Gamma \Rightarrow A \vdash}$$

# 1-categorical semantics of SOGATs

## Rough strategy

- for each signature  $\Sigma$ ,
  - a locally presentable category  $\llbracket \Sigma \rrbracket$  and
  - a model of type theory  $\text{ULF}_\Sigma$  with dependent products;
- interpret signature extension rules.

In fact:

- $\llbracket \Sigma \rrbracket$  consists of a functor  $\pi^\Sigma: |\Sigma| \rightarrow \mathbf{Cat}_1^\ddagger$ ;
- here, “model of type theory” means **generalised CwF**.

## Notation

Omitting  $|-|$ .

---

$\ddagger$ Cats with terminal object.

# Gecawfs (essentially Coraglia, Di Liberti, Emmenegger)

Let  $\mathcal{K}$  be any 2-category.

## Generalised category with families (gecawf) in $\mathcal{K}$

$$\begin{array}{ccc}
 & \text{var} & \\
 & \curvearrowright & \\
 tm & \xrightarrow{\tau} & ty \\
 & \text{proj} & \\
 \pi^{tm} \searrow & & \swarrow \pi^{ty} \\
 & C & 
 \end{array}$$

where

- $\pi^{ty}$  and  $\pi^{tm}$ : fibrations in  $\mathcal{K}$ ,
- $proj$ : fibration morphism,
- components of unit and counit are cartesian.

# The main gecawf

$$\begin{array}{ccc}
 \mathbf{SPreNatMod} & \xrightarrow{\partial} & \mathbf{PreNatMod} \\
 & \searrow & \swarrow \text{cod}_v \\
 & & \mathbf{DFib}
 \end{array}$$

- Right adjoint: self pullback.
- $\leadsto$  **context extension** is  $dom_v : \mathbf{PreNatMod} \rightarrow \mathbf{DFib}$ .

# Interpretation of $ULF_{\Sigma}$

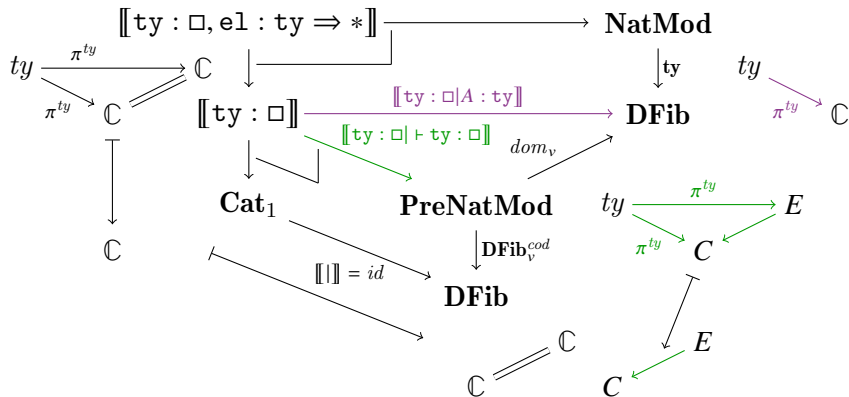
Taken care of by “homming into” the main gecawf:

$$\begin{array}{ccc}
 [\Sigma, \mathbf{SPreNatMod}] & \xrightarrow{[\Sigma, \partial]} & [\Sigma, \mathbf{PreNatMod}] \\
 & \searrow & \swarrow [\Sigma, \text{cod}_v] \\
 & & [\Sigma, \mathbf{DFib}]
 \end{array}$$

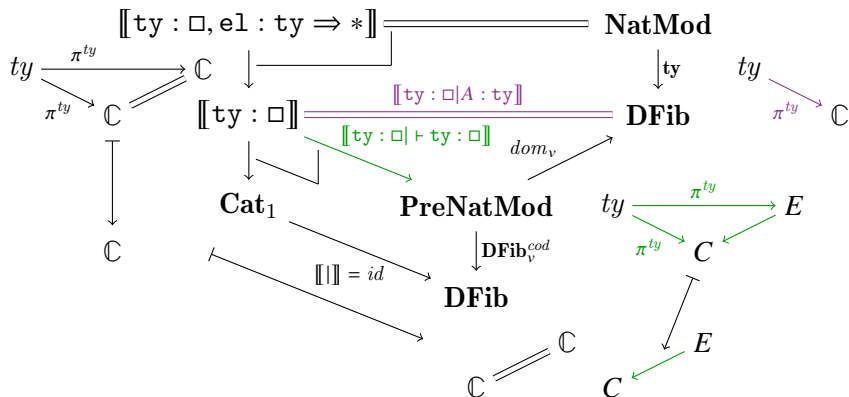
## Claim

Each  $\llbracket ULF_{\Sigma} \rrbracket$  has dependent products.

# Application



# Application





# Application

Signature for (object) dependent product type:

$$\Sigma_{\Pi} := (\text{ty} : \square, \text{el} : \text{ty} \Rightarrow *, \Pi : (A : \text{ty}, B : \text{el}(A) \rightarrow \text{ty}) \Rightarrow \text{ty}).$$

Interpretation of input arity  $\dots | A : \text{ty} \vdash \text{el}(A) \rightarrow \text{ty} : \square$ :

$$\begin{array}{ccccc}
 \llbracket \text{el}(A) \rightarrow \text{ty} \rrbracket & \rightarrow & \text{tm} \times_{\mathbb{C}} \text{ty} & \longrightarrow & \text{ty} \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \pi^{\text{ty}} \\
 \text{ty} & \xrightarrow{\text{var}} & \text{tm} & \xrightarrow{\pi^{\text{tm}}} & \mathbb{C} \\
 & \searrow \text{ext} & & & 
 \end{array}$$

i.e.,

$$\gamma \mapsto \coprod_{a \in \text{ty}_{\gamma}} \text{ty}_{\text{ext}(\gamma, a)}.$$