

Path Types  
in  
Algebraic Type Theory

Steve Awodey  
jww/ Joseph Hua

SSTT 3  
Ljubljana  
June 2026

# Outline

- 1) Review of Natural Models
- 2) Path Types
- 3) Examples
- 4) Cubical Kan Structure

# 1. Natural Models

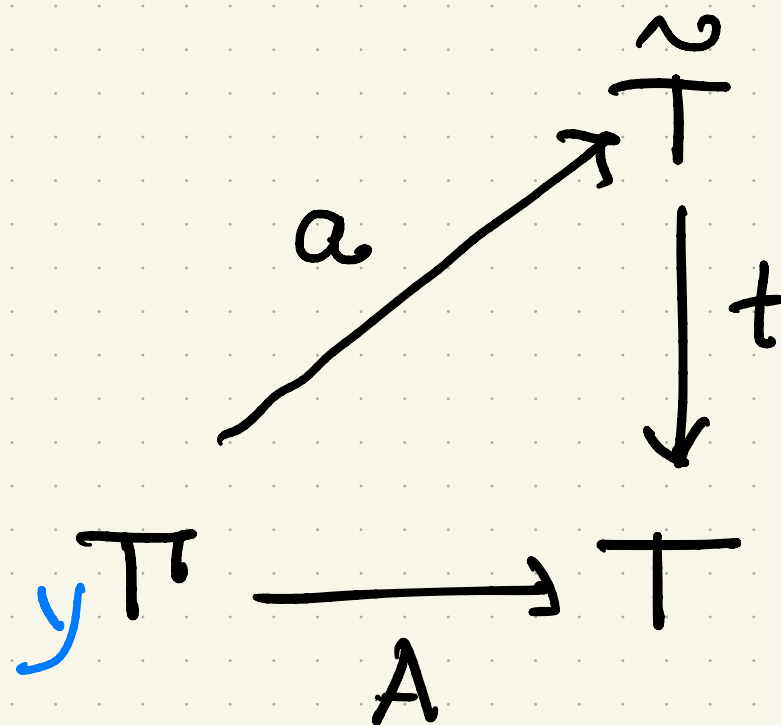
11

Def. A natural model consists of:

- a cat  $\mathcal{C}$  of contexts  $\sigma: \Delta \rightarrow \top$ ,
- presheaves  $\mathbb{T}, \tilde{\mathbb{T}}$  of types & terms,
- a natural transformation  $t: \tilde{\mathbb{T}} \rightarrow \mathbb{T}$  that types the terms,

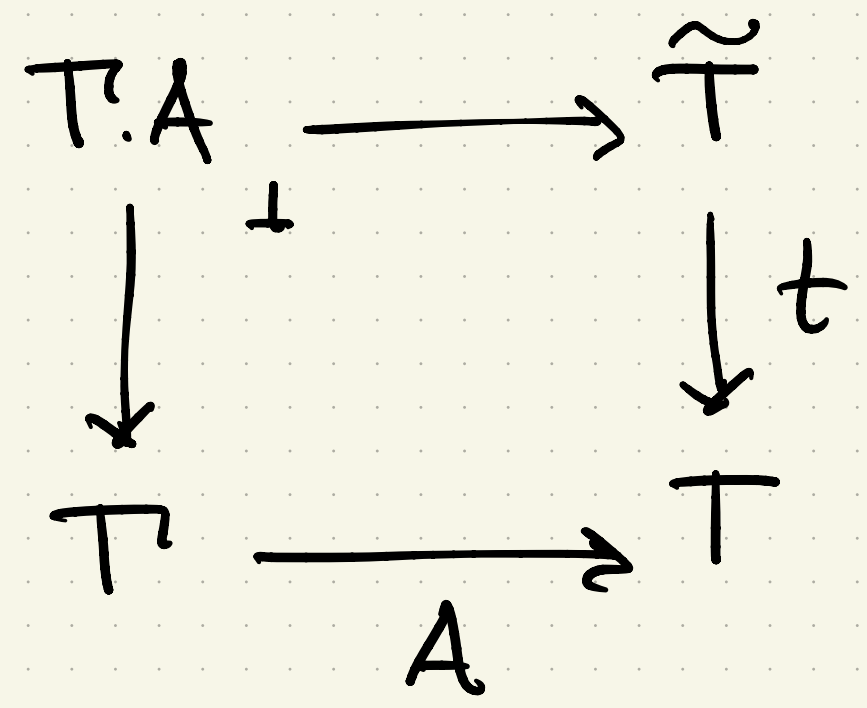
type theory

$$\top \vdash a : A$$



10

•  $t: \tilde{T} \rightarrow T$  is representable:



this models context extension,

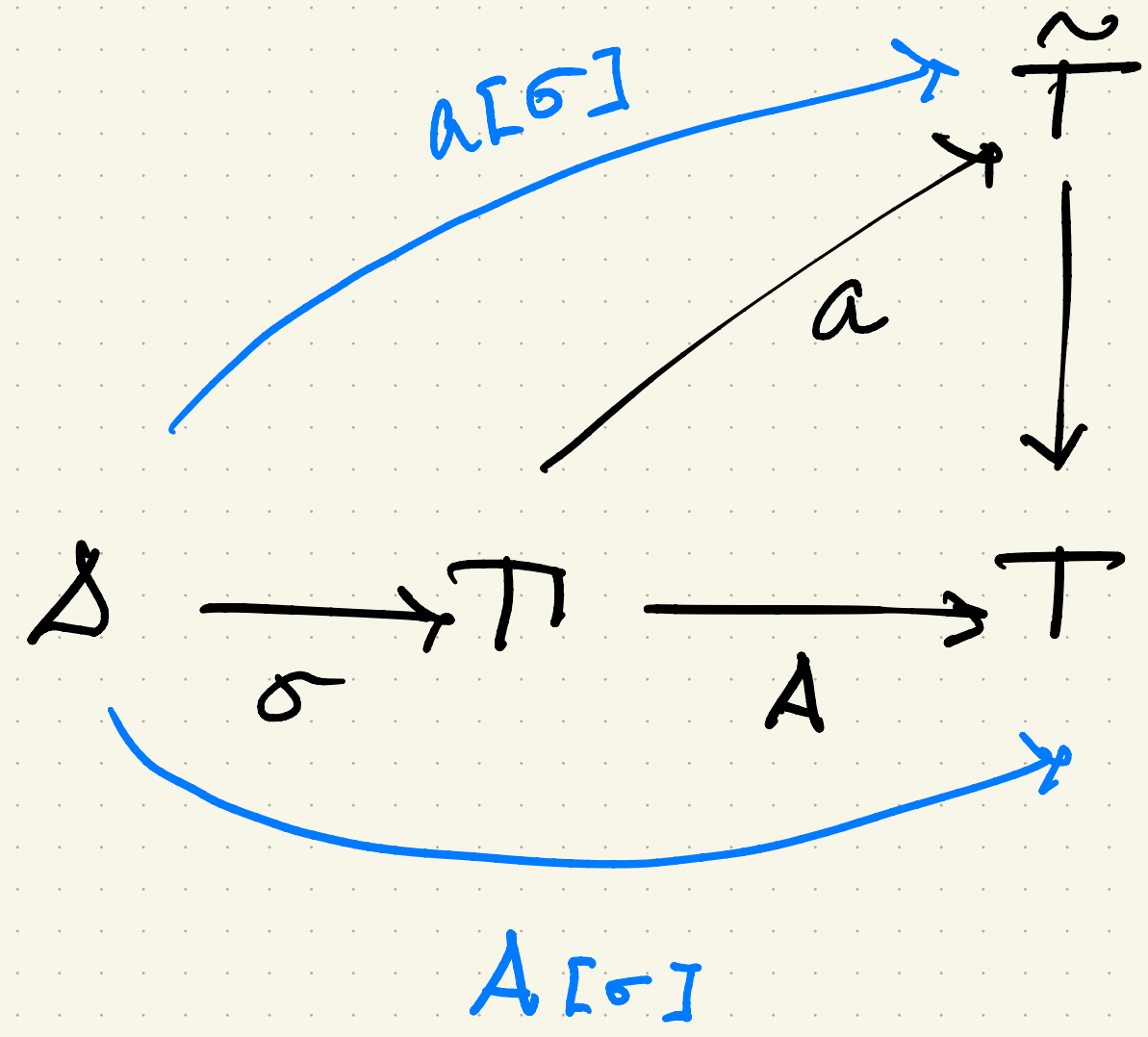
$$\frac{T \vdash A}{T.A \vdash}$$

$$\pi_A: T.A \rightarrow T$$

- Substitution is modelled by composition\*

$$\frac{\Pi \vdash a : A \quad \sigma : \Delta \rightarrow \Pi}{\Delta \vdash a[\sigma] : A[\sigma]}$$

$$\Delta \vdash a[\sigma] : A[\sigma]$$

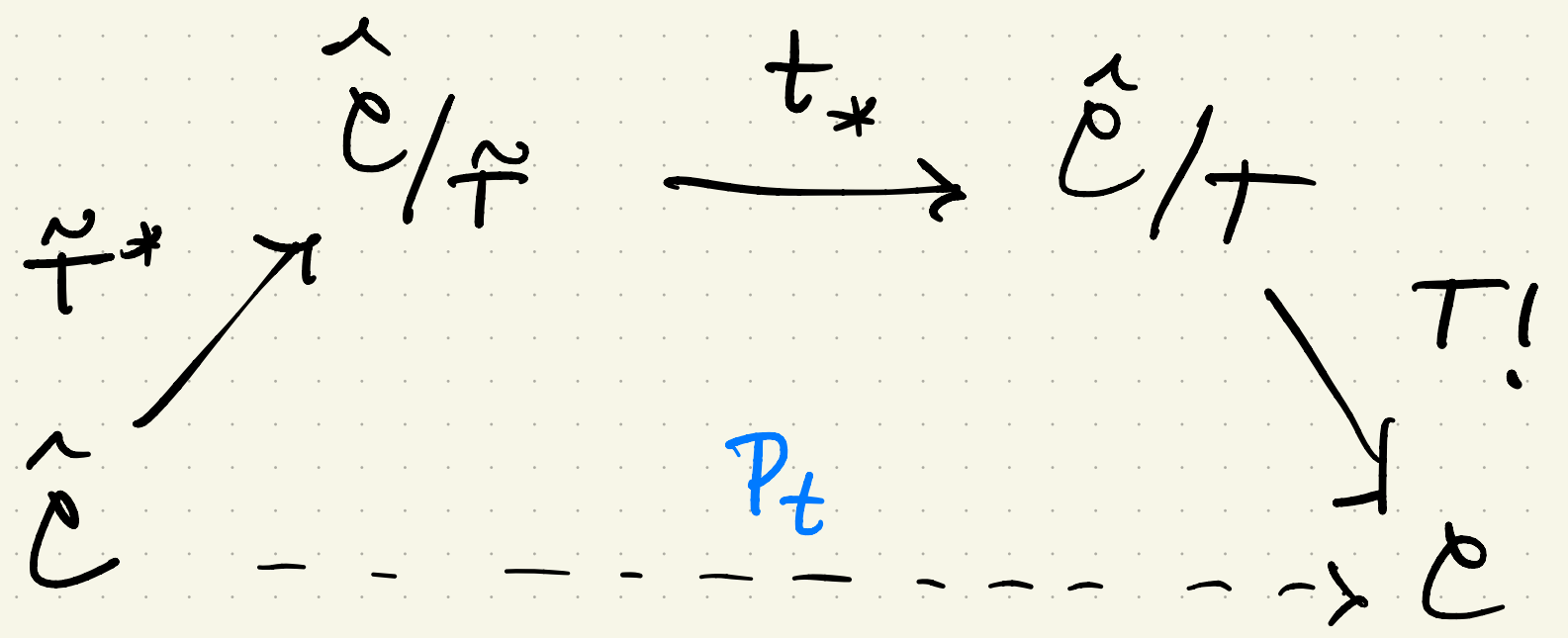


\* ) not pullback!

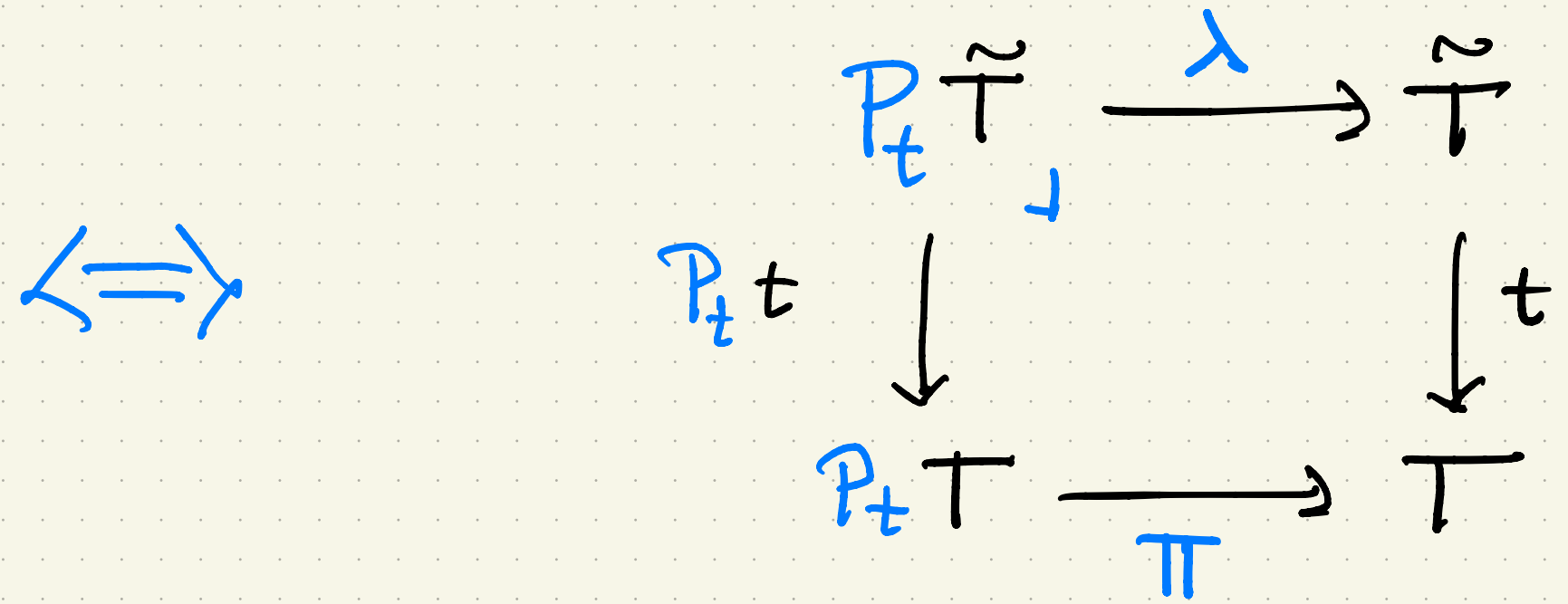
- Type formers  $\Sigma, \Pi$  are modelled using the polynomial endofunctor of  $t: \tilde{T} \rightarrow T$ :

$$P_t: \hat{\mathcal{C}} \rightarrow \hat{\mathcal{C}}$$

$$P_t(X) = \sum_{A:T} X^A$$



• E.g.  $\Pi$ -Rules of Form - Intro - Elim - Comp



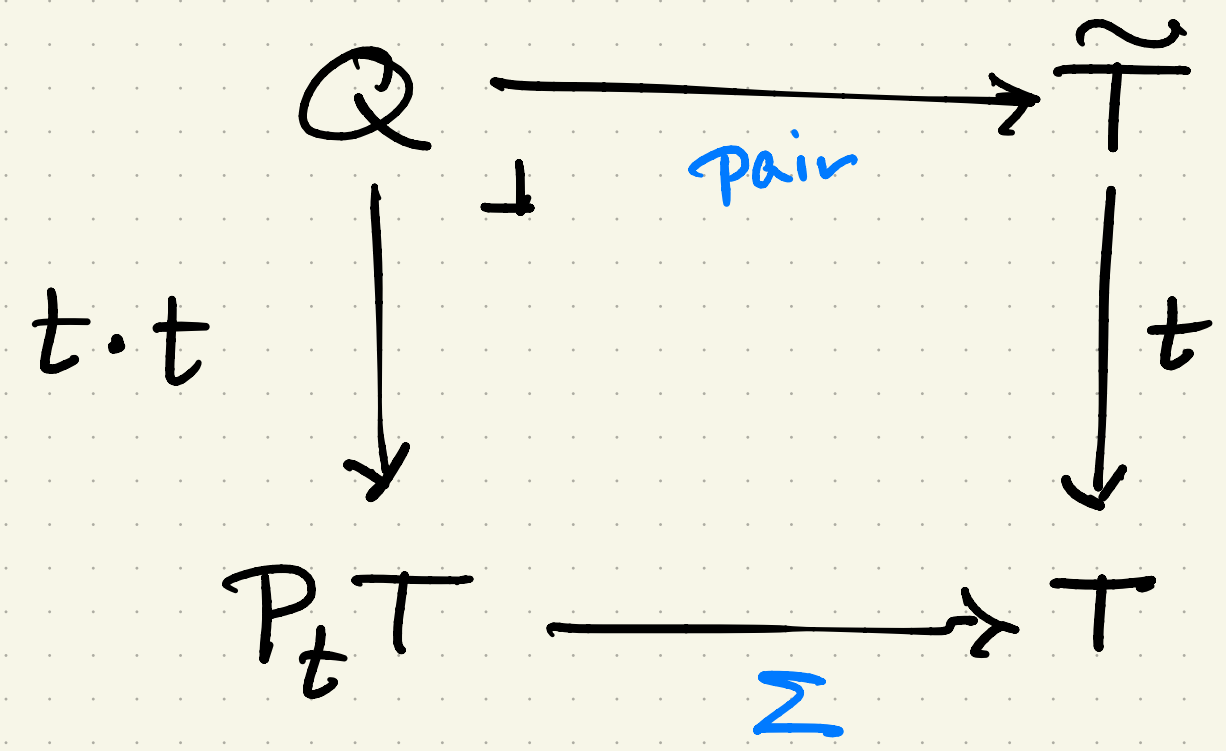
$$\text{Form} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash \Pi_A B}$$

$$\frac{\Gamma, A \vdash b : B}{\Gamma \vdash \lambda_A b : \Pi_A B} \quad \text{Intro}$$

• The  $\Sigma$ -Rules state a multiplication

$$(\Sigma, \text{pair}) : P_t \circ P_t \implies P_t$$

in the cat  $\text{Poly}(\hat{\mathcal{E}})$  of polynomial functors,



where

$$P_{t.t} = P_t \circ P_t .$$



## 2. Path Types

(8)

Now fix an interval in  $\hat{\mathcal{C}}$ ,

$$1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} I .$$

For each object  $X$  we have a pathobject,

$$X \longrightarrow X^I \rightrightarrows X$$

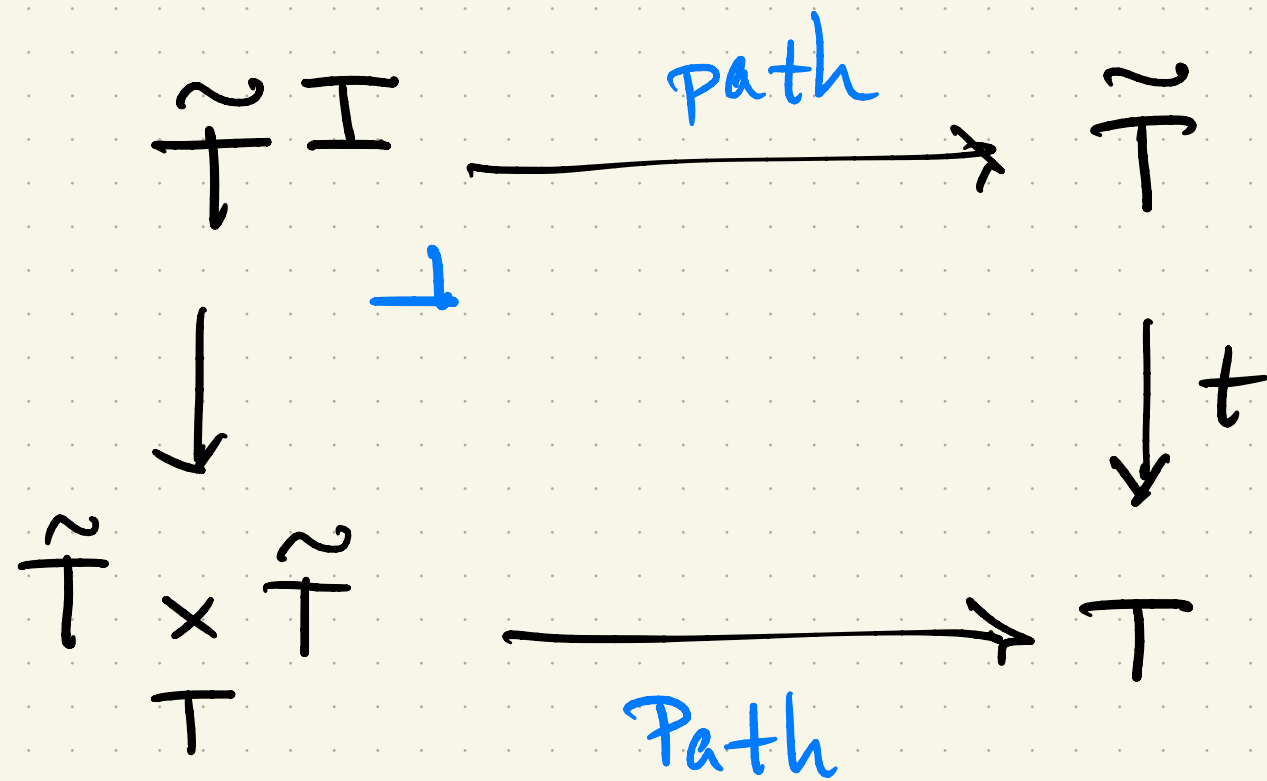
and a factorization

of  $\Delta_X$ .

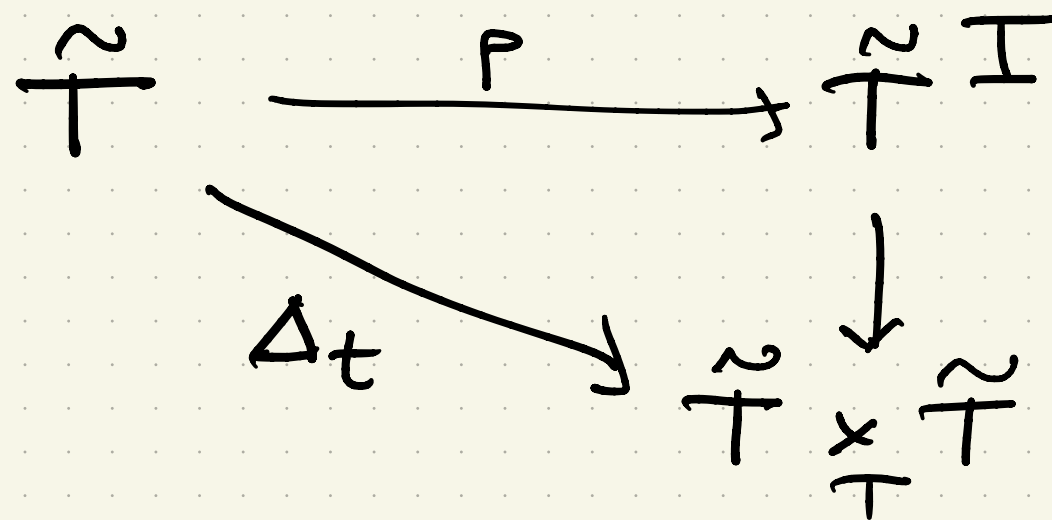
$$\begin{array}{ccc} X & \xrightarrow{e} & X^I \\ & \searrow \Delta_X & \downarrow \\ & & X \times X \end{array} .$$

Def.  $t: \tilde{T} \rightarrow T$  has path types if

there are maps:



where

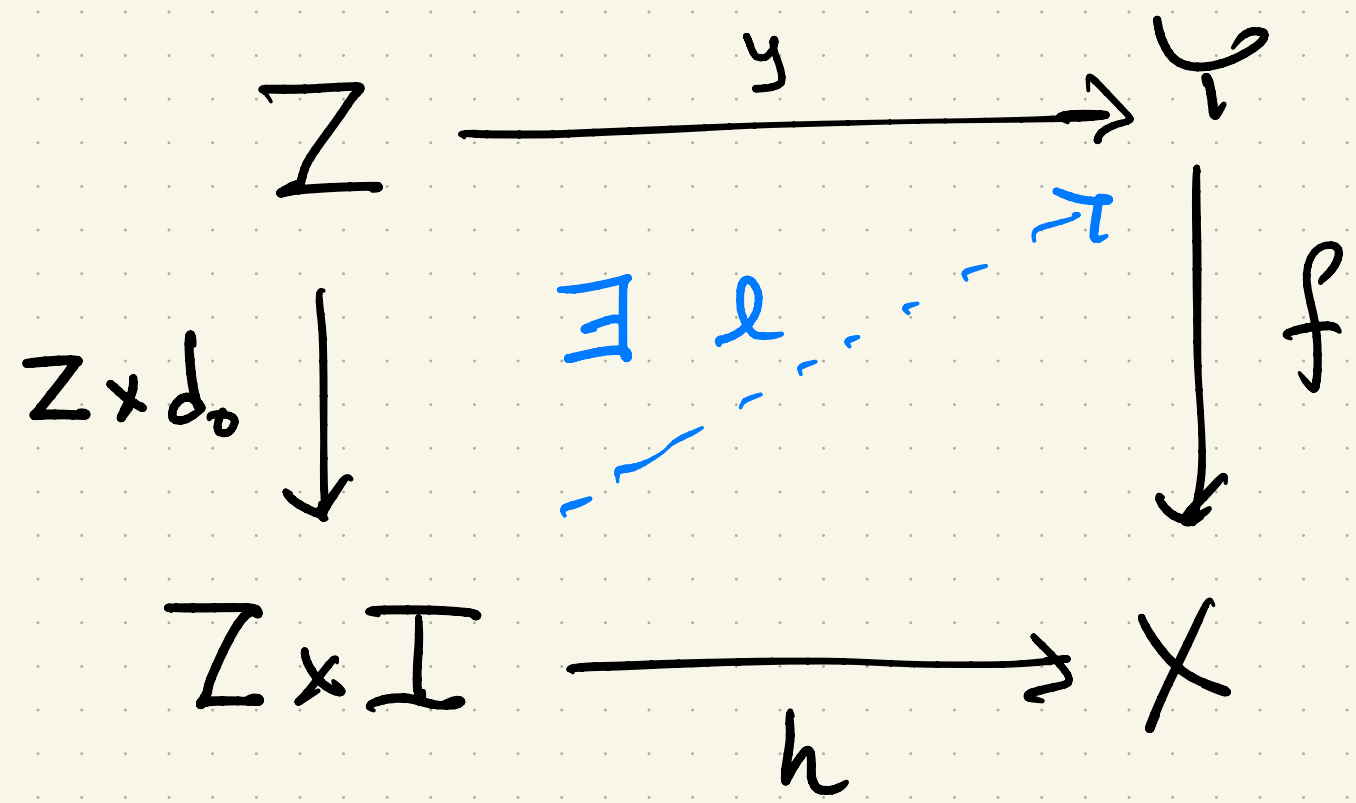


is the path object factorization of  $\Delta_t$  over  $T$ .

We need the following:

Def.  $f: Y \rightarrow X$  is a Hurewicz fibration if

for all  $Z$  &  $y$  &  $h$ ,

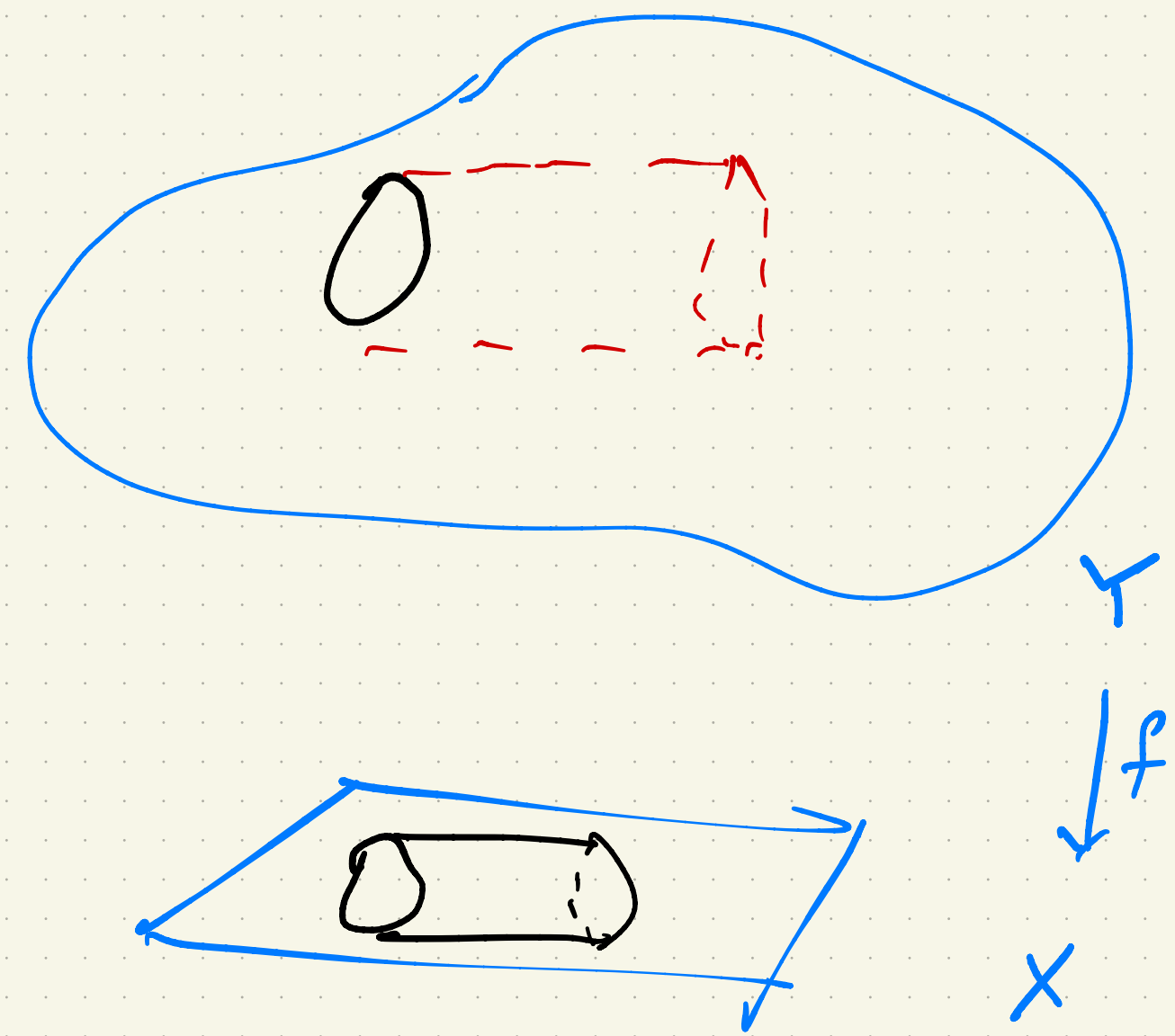
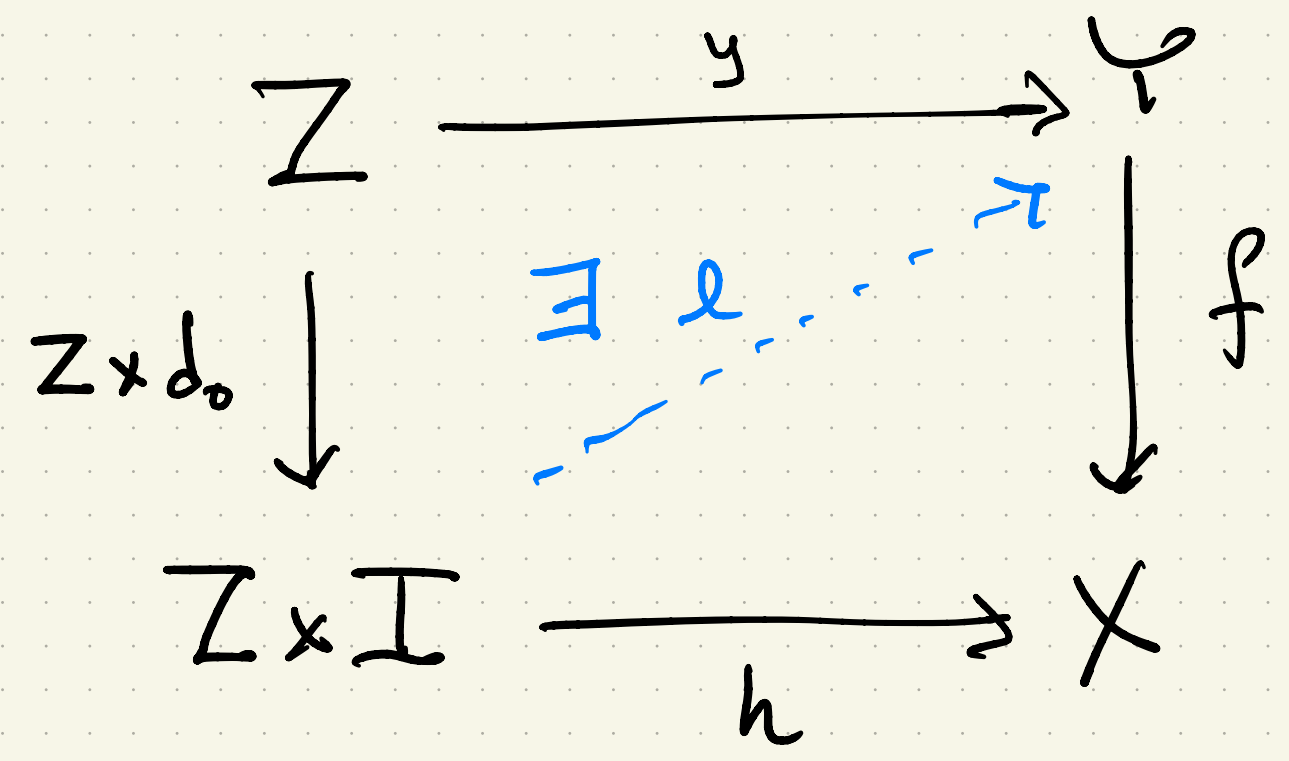


"Homotopy lifting property"

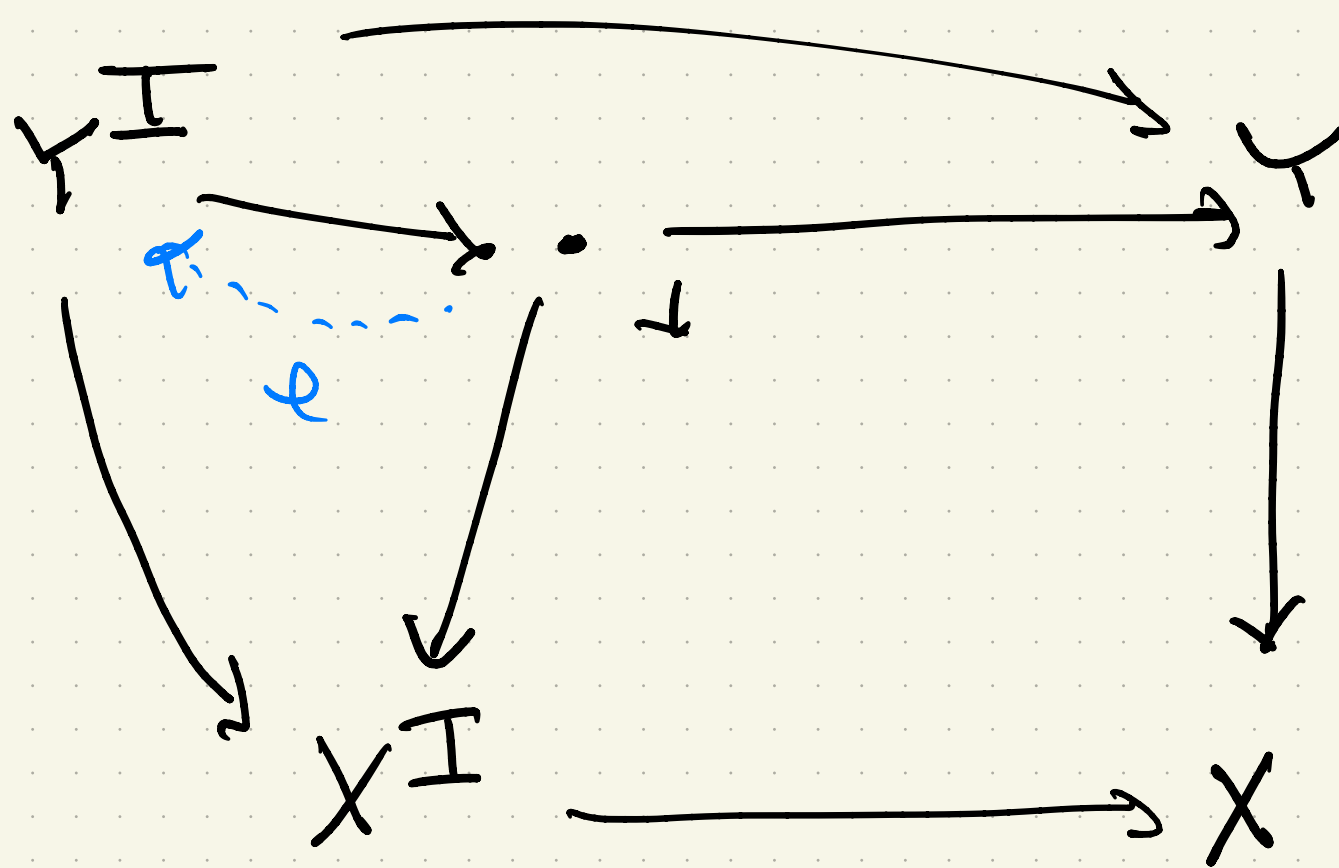
We need the following:

Def.  $f: Y \rightarrow X$  is a Hurewicz fibration if

for all  $Z$  &  $y$  &  $h$ ,



Fact If  $f: Y \rightarrow X$  is Hurewicz, then  
 there's a section  $l: X^I \times_X Y \rightarrow Y^I$ :

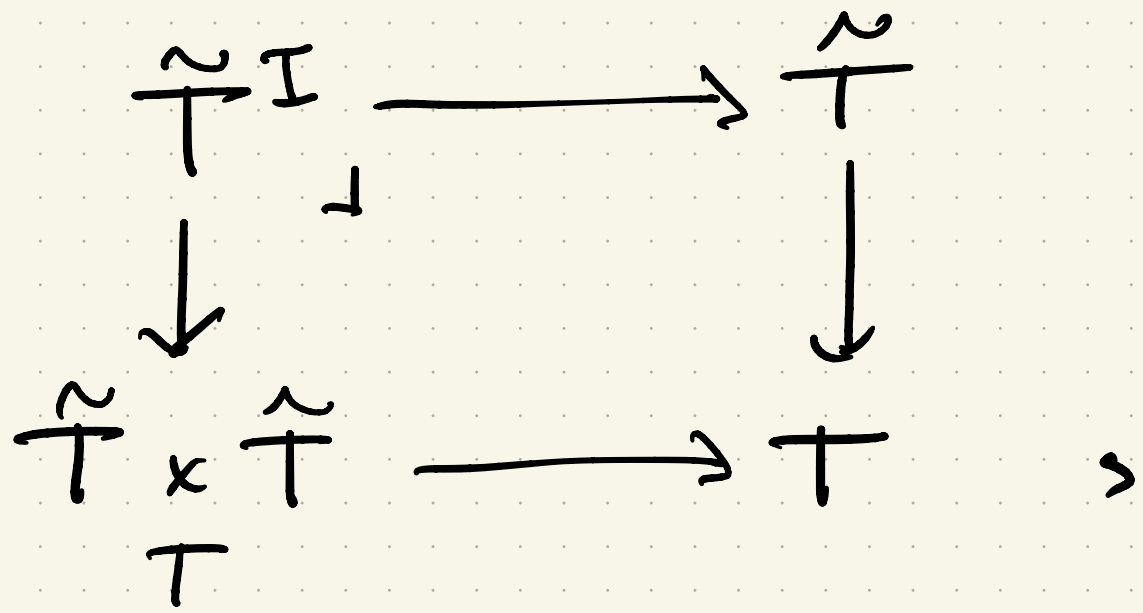


Call  $f$  normal if  $l(p, -) = p$ .

Prop.

Suppose  $t: \tilde{T} \rightarrow T$ ,

i) has path types



ii) is a normal Hurewicz fibration.

Then  $t$  models intensional Id-types.

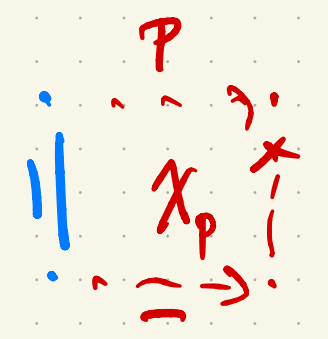
Lemma. If  $t: \tilde{T} \rightarrow T$  satisfies (i) & (ii),

then any type  $A \rightarrow X$  classified by  $t$

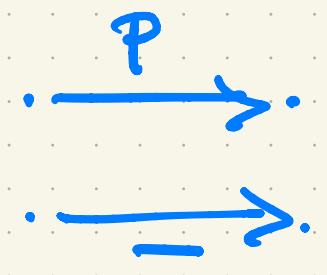
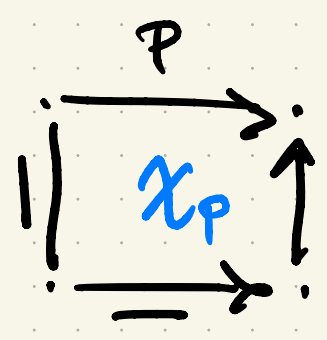
has a connection  $\chi: A^I \rightarrow A^{I \times I}$ .

pf.

$$A^I \xrightarrow{\chi} A^{I \times I}$$

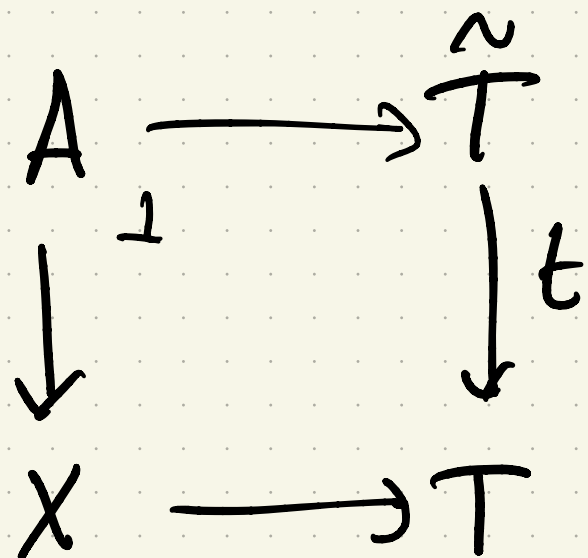


$$\cdot \xrightarrow{p} \cdot$$

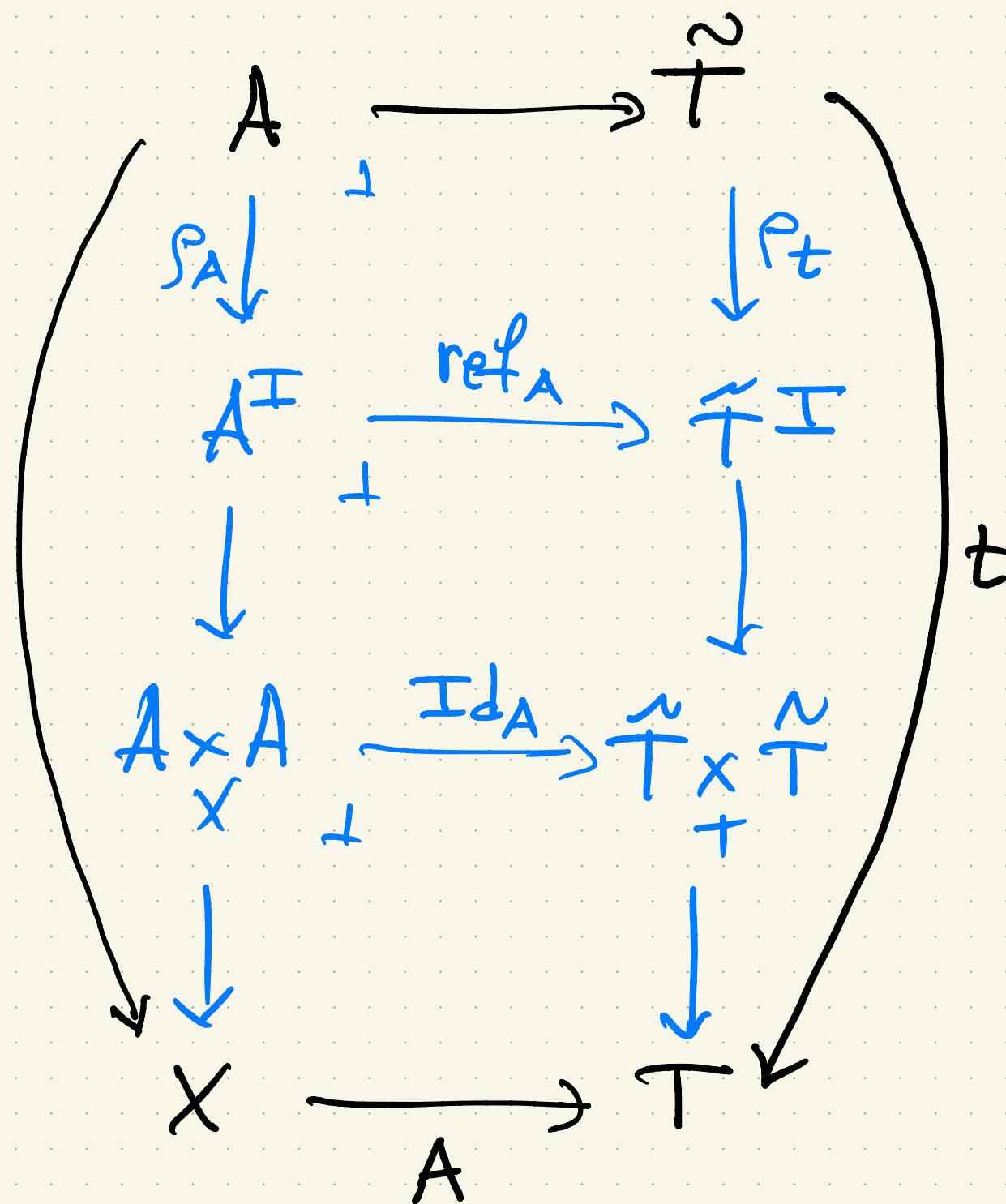


# Pf. of Prop

1) Given:



2) We have:



3) Models:

$X \vDash A$

Form

$X, x, y: A \vdash \text{Id}_A(x, y)$

•/•

Intro

$X, x: A \vdash \text{ref}_A(x) : \text{Id}_A(x, x)$

4) NTS:

$$x, y: A, p: Id_A(x, y) \vdash C$$

$$x: A \vdash c: C_p$$

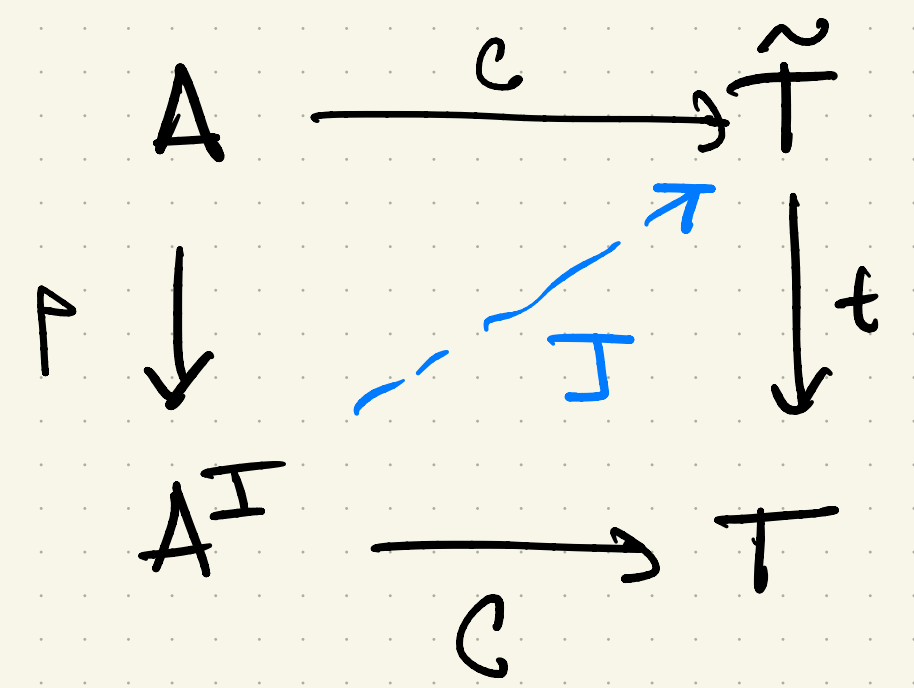
Elim

---

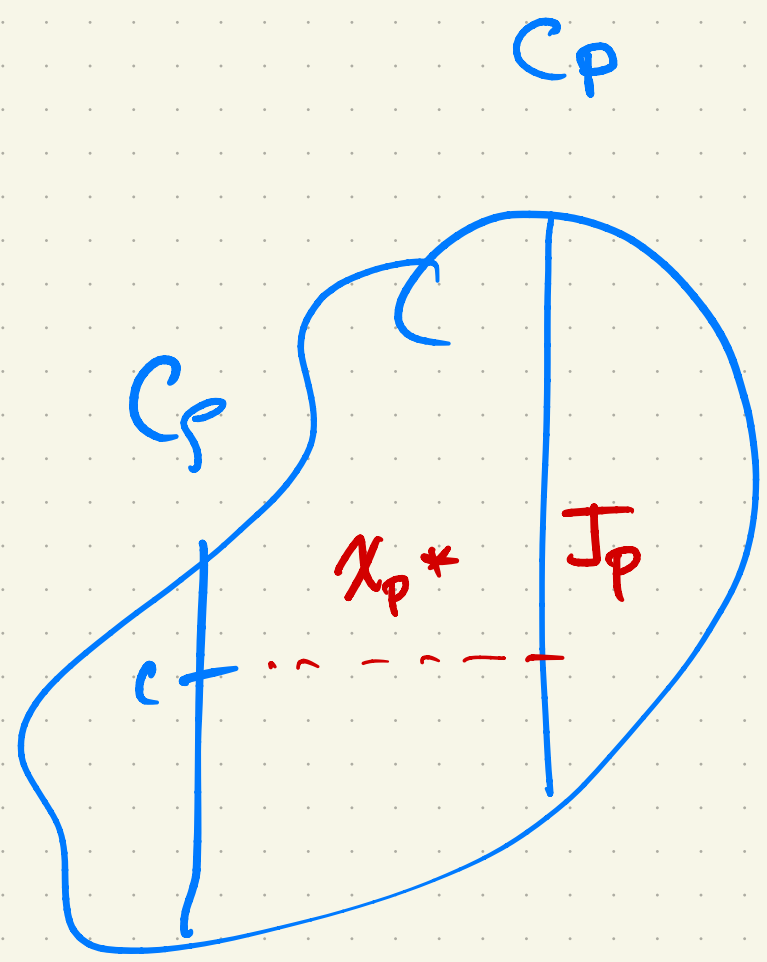
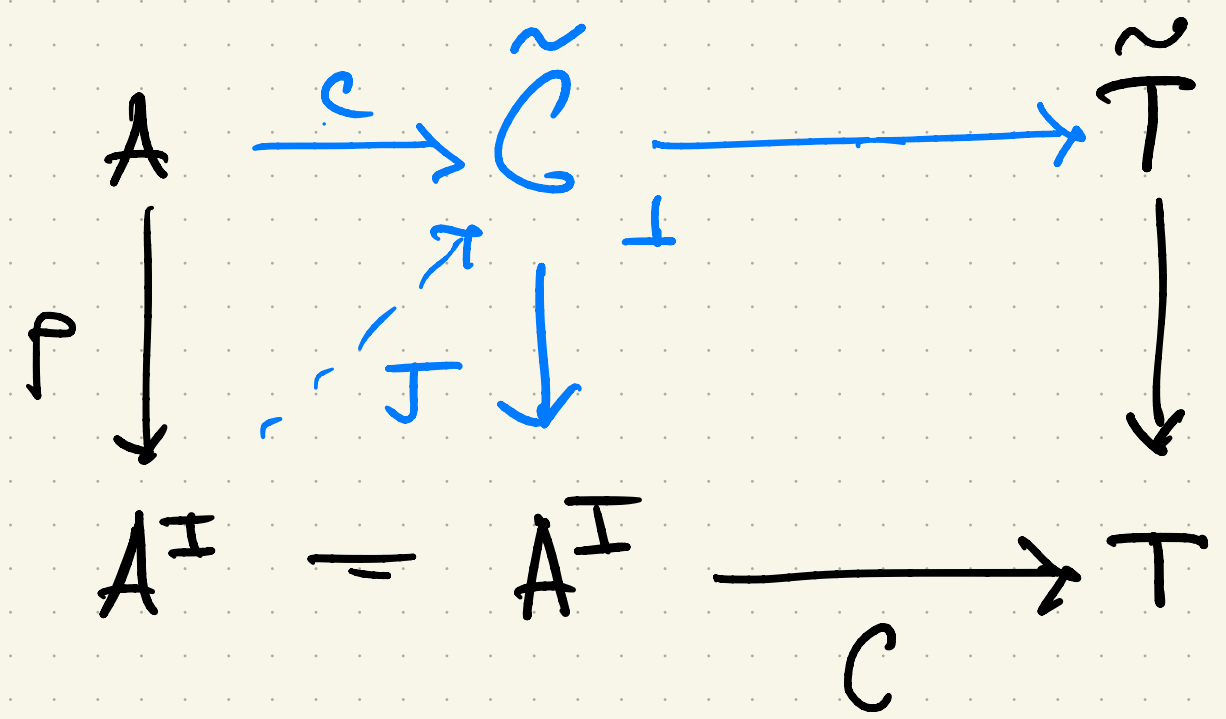
$$x, y: A, p: Id_A(x, y) \vdash J: C$$

Comp

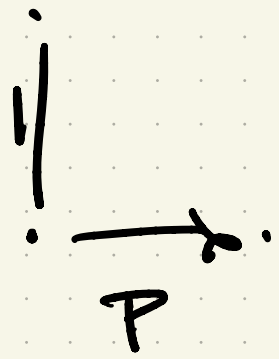
$$\frac{\%}{x: A \vdash J_p \equiv c}$$



Equivalently :

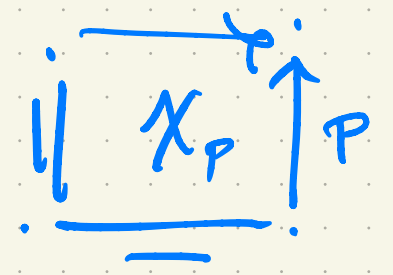


5) Take  $\quad$  and  $\quad$  get



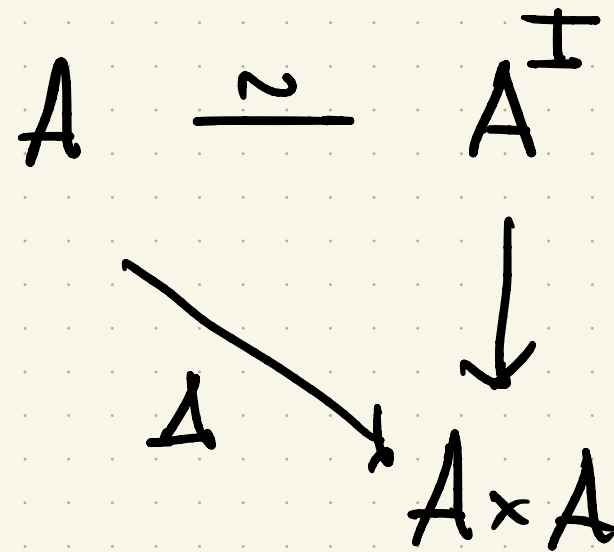
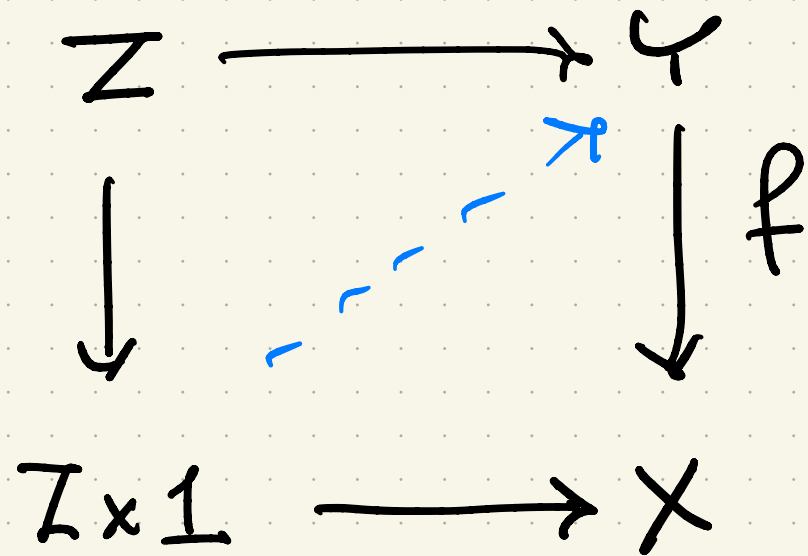
$c: C_p$

$x_p^* + c =: J_p$



# 3. Examples

1)  $I = \underline{1}$  in any LCC:

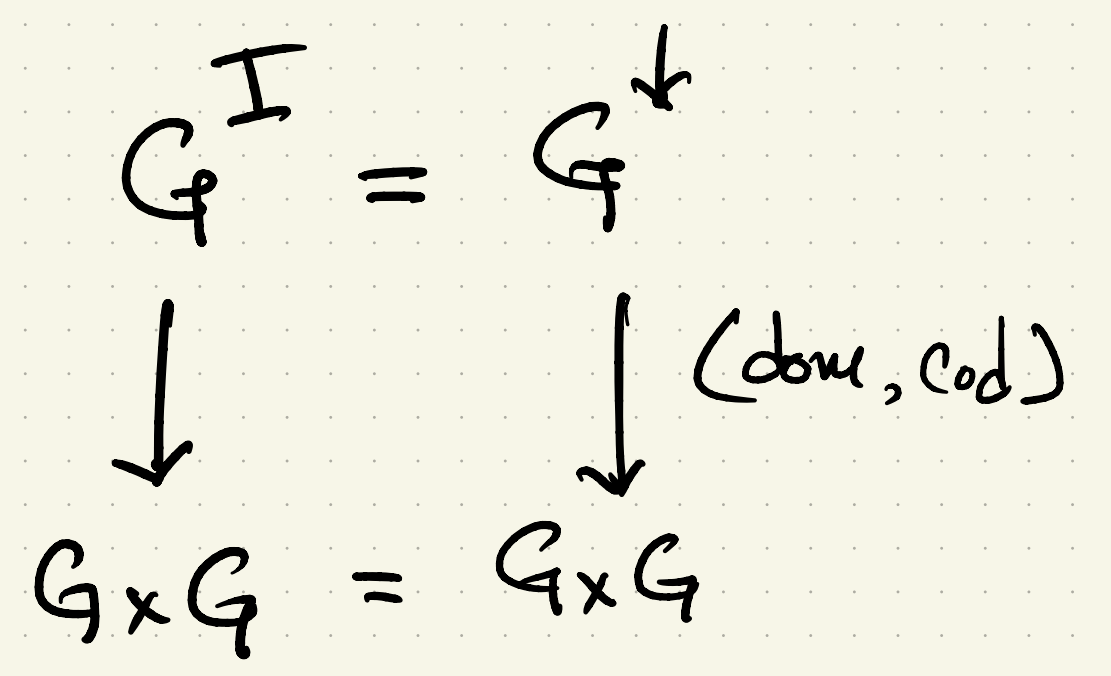
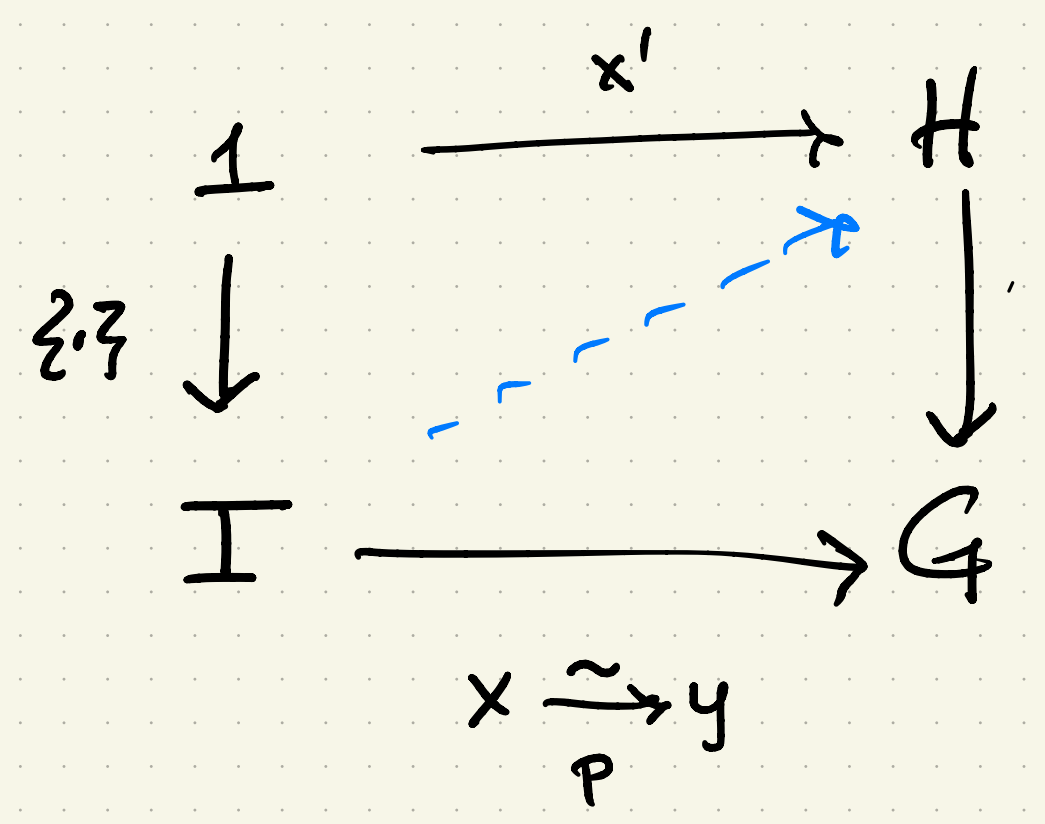


all  $f: Y \rightarrow X$   
are H-fibrations

path types  
are diagonals

Extensional TT

2)  $I = \cdot \xrightarrow{\sim} \cdot$  in Gpds:



Hurewicz  
= (iso-)fibration

Path object  
= arrow groupoid

Usual H-S Groupoid model

3)  $I = [1]$  in  $s\text{Set} = \text{Set}^{\Delta^{\text{op}}}$  :

20

Kan fibrations are Hurewicz,  
so Id.types follow from the Prop.

4)  $I = [1]$  in  $c\text{Set} = \text{Set}^{\#^{\text{op}}}$  :

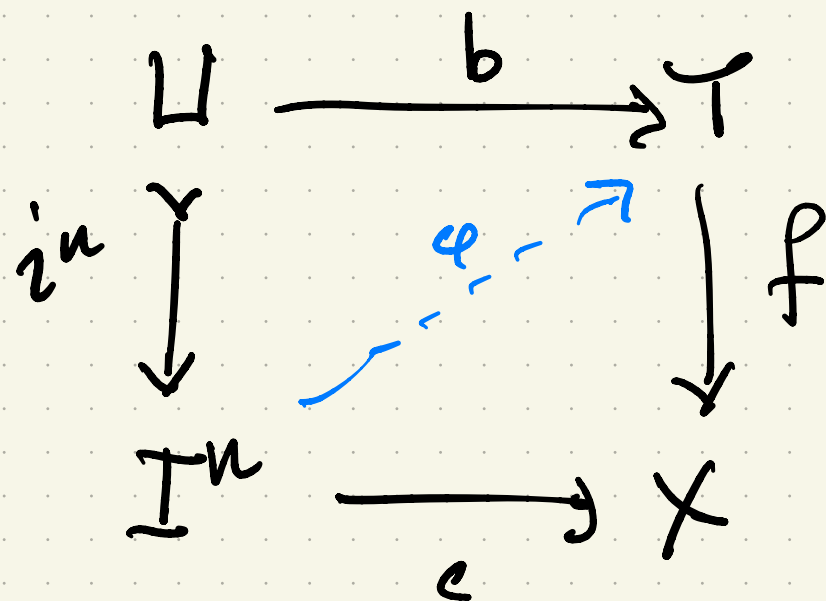
Hurewicz + Path types

$\Rightarrow$  n-box filling f. all  $n > 0$ !

# 4. Box Filling

21

Def.  $f: Y \rightarrow X$  has n-box filling if for all open boxes  $i^n: U \rightarrow I^n$ , and all  $b, c$ , there's a diagonal filler  $\varphi$ , as in:



Briefly,

$$i^n \dashv f$$

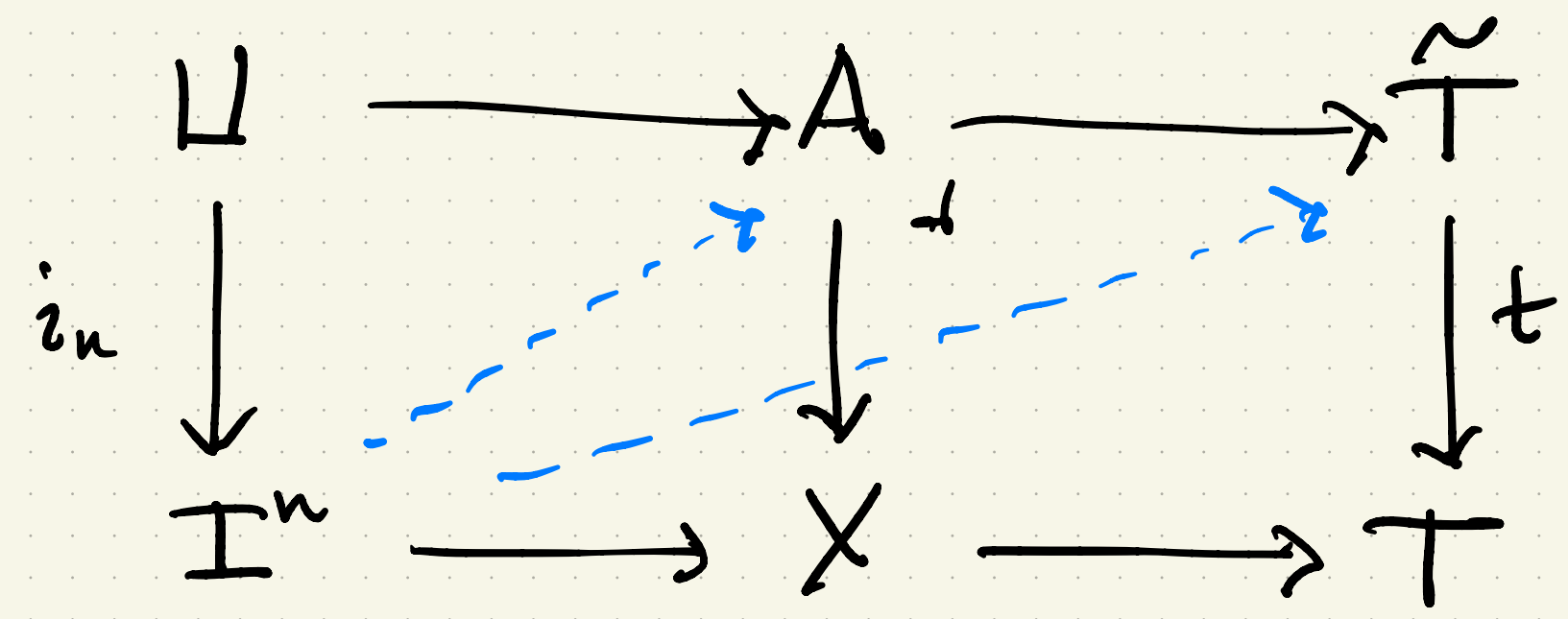
Prop. Suppose the model  $t: \tilde{T} \rightarrow T$

(i) has path types,

(ii) is Hurewicz.

Then any type family  $A \rightarrow X$  classified by  $t$  has  $n$ -box filling for all  $n > 0$ .

Pf.



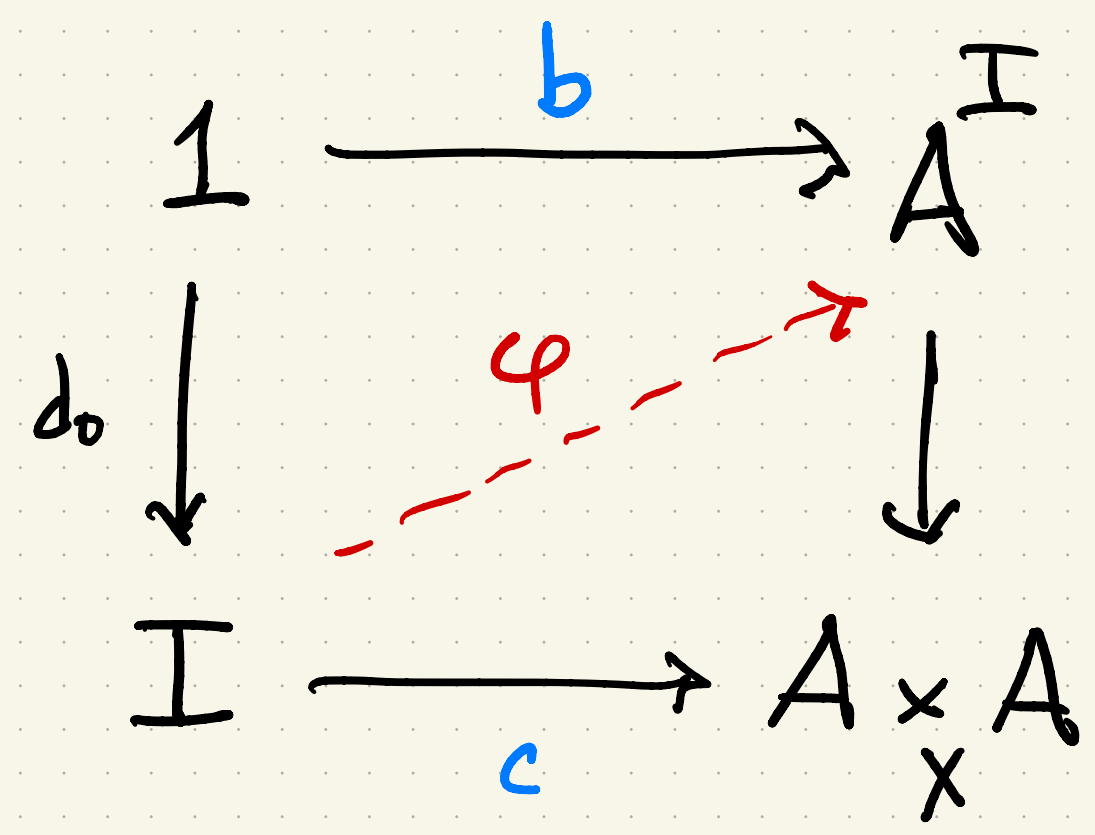
STS:

$i_n \uparrow t$   
 $\forall n > 0$

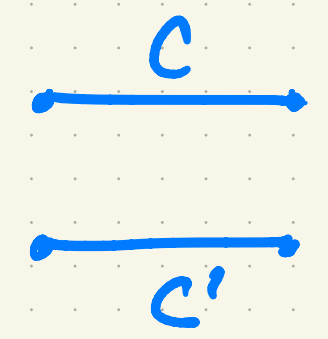
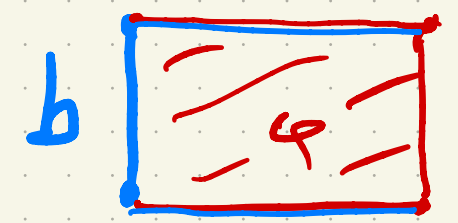
Lemma For any  $A \rightarrow X$ ,

$$i_n \dashv A^I \iff i_{n+1} \dashv A$$

Pf.



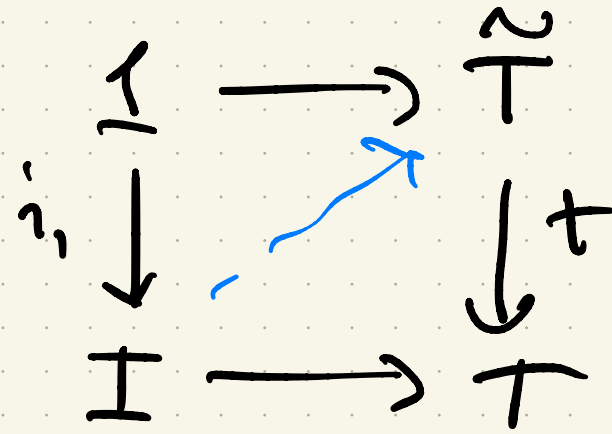
$$i_1 \dashv A^I$$



$$i_2 \dashv A$$

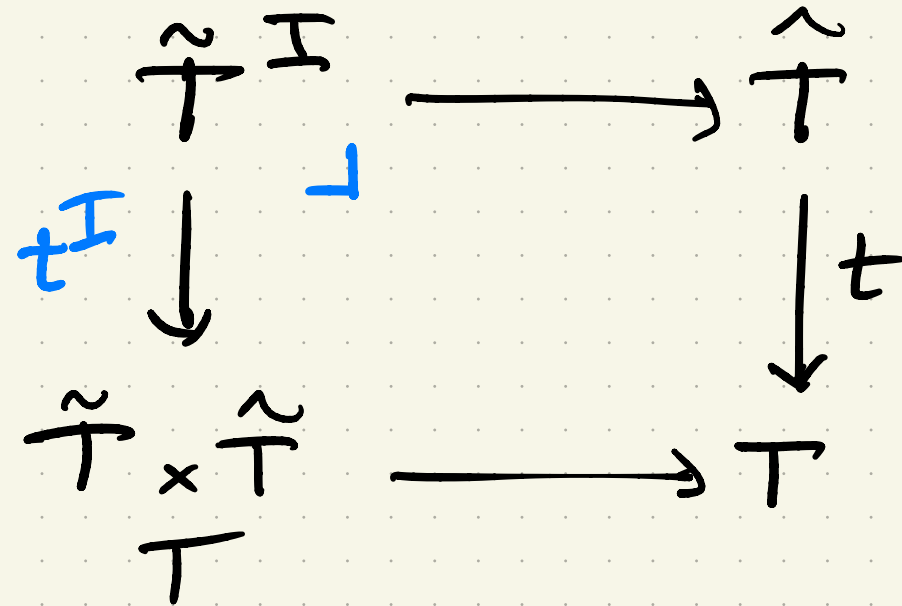
Pf. of Prop.

1)  $t$  Hurewicz  $\Rightarrow$



$i_1 \circ t$ ,

2) path types  $\Rightarrow$

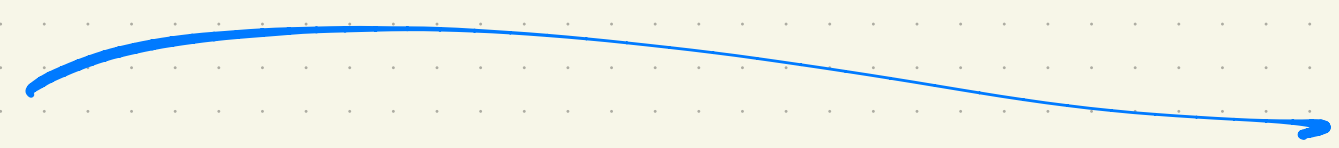


$i_1 \circ t^{\mathbb{H}}$ ,

3) Lemma  $\Rightarrow i_2 \circ t$ ,

Etc., by induction on  $n > 0$ .

THANKS!



Reference:

S.A. & J. Hua : Path Types in ATT,  
arXiv, 2026 .