

A Type Theory for Comprehension Categories

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- *From Semantics to Syntax: A Type Theory for Comprehension Categories* [Naj+26], POPL 2026
- Ongoing work in its continuation

- Martin-Löf type theory (MLTT) serves as a foundation for proof assistants and programming languages
- Several well-established categorical semantics: contextual categories, categories with families, display map categories, natural models, . . .
- Comprehension categories as a general framework to organize these notions

[ALN24]:

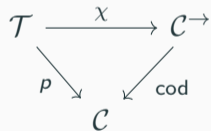
“We take comprehension categories as a unifying language and show how almost all established notions of model embed as sub-2-categories (usually full) of the 2-category of comprehension categories.”

Motivation

Interpretation of MLTT in comprehension categories:



Semantics does not use all the features of



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$$\text{MLTT} \xrightarrow{\text{semantics}} \begin{array}{ccc} \mathcal{T} & \xrightarrow{\chi} & \mathcal{C}^{\rightarrow} \\ & \searrow p & \swarrow \text{cod} \\ & \mathcal{C} & \end{array}$$

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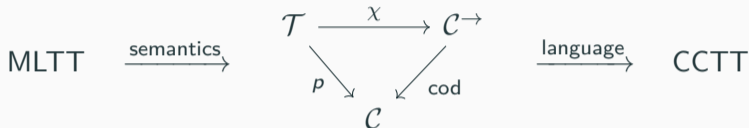
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Two options:

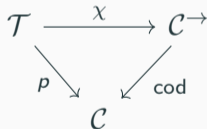
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Interpretation of MLTT in comprehension categories:



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Two options:

1. Restrict the comprehension categories to 'simple' ones (fully faithful or discrete)
2. **Make the type theory more expressive: CCTT**

Why not Restrict the Models

- Are there interesting examples we would miss?
- Are there interesting features that we would lose?

More on this after some preliminaries

1. Review: Comprehension Categories
2. Back to Our Motivation
3. Our Work: Core Syntax CCTT
4. CCTT Captures Subtyping
5. Extending CCTT with Type Formers
6. Conclusion and Current Work

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Comprehension Categories

Comprehension Category [Jac93]

1. a category \mathcal{C} ,
2. a (cloven) fibration $p : \mathcal{T} \rightarrow \mathcal{C}$,
3. a functor $\chi : \mathcal{T} \rightarrow \mathcal{C}^{\rightarrow}$ preserving cartesian arrows,

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\chi} & \mathcal{C}^{\rightarrow} \\ & \searrow p & \swarrow \text{cod} \\ & \mathcal{C} & \end{array}$$

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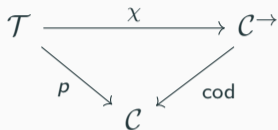
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A comprehension category is

- full** if χ is full and faithful;
- split** if p is a split fibration.

Interpreting MLTT in a Comprehension Category



Category \mathcal{C} models contexts

Fibre \mathcal{T}_{Γ} models types in context Γ

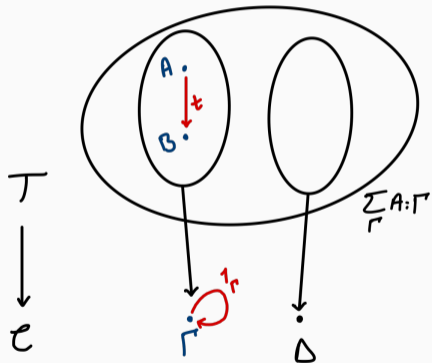
Reindexing models substitution

Comprehension χ models context extension $(\Gamma, A) \mapsto (\Gamma.A \xrightarrow{\chi(A)} \Gamma)$

Sections of $\chi(A)$ model terms $\Gamma \vdash t : A$

Vertical Morphisms

What about **morphisms in a fibre** \mathcal{T}_Γ ?

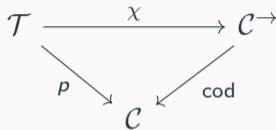


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Before we wondered:

- Are there interesting examples we would miss?
- Are there interesting features that we would lose?

Syntactical Features From Non-Full Comprehension Categories



- A comprehension category can express both
 1. morphisms between contexts and
 2. morphisms between types.
- Interpreting MLTT does not make use of morphisms of types, hence these are often taken to be trivial (\mathcal{T} discrete) or coming from \mathcal{C} (χ fully faithful)
- Restriction 'kills off' this 'extra dimension' of morphisms.

Later we will see that this extra dimension captures **coercive subtyping**.

Examples of Non-Full Comprehension Categories I

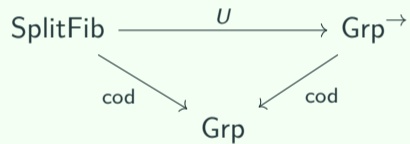
- Intensional type theories are often given semantics in an algebraic weak factorisation system (AWFS)
- AWFSs¹ give rise to **non-full** comprehension categories

$$\begin{array}{ccc} \text{EM}(R) & \xrightarrow{U} & \mathcal{C}^{\rightarrow} \\ & \searrow & \swarrow \text{cod} \\ & \mathcal{C} & \end{array}$$

We capture this extra semantic structure in CCTT.

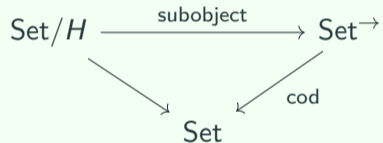
¹with \mathcal{C} that has finite limits

Example (Groupoids)



Examples of Non-Full Comprehension Categories III

Example (Heyting algebra H)



χ maps every $p : X \rightarrow H$ to the set $\{x \in \Gamma \mid \top \leq p(x)\}$.

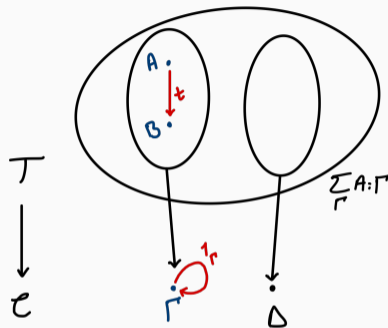
1. We design rules of a type theory that reflect the structure of comprehension categories: CCTT
2. We show how some rules of CCTT can be seen as rules for coercive subtyping, extending work by Coraglia and Emmenegger [CE24]
3. Extend CCTT with Π -, Σ - and Id-types and their compatibility with subtyping

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1. $\Gamma \text{ ctx}$
2. $\Gamma \vdash s : \Delta$
3. $\Gamma \vdash s \equiv s' : \Delta$
4. $\Gamma \vdash A \text{ type}$
5. $\Gamma | A \vdash t : B$
6. $\Gamma | A \vdash t \equiv t' : B$

CCTT: Judgements

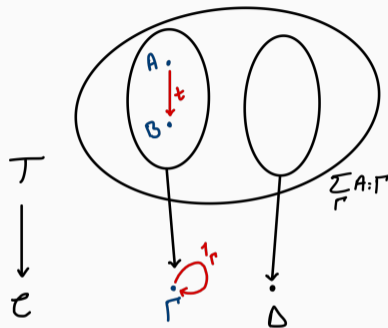
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- Judgement 5: a morphism $\llbracket t \rrbracket : \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket$ in the fibre $\mathcal{T}_{\llbracket \Gamma \rrbracket}$

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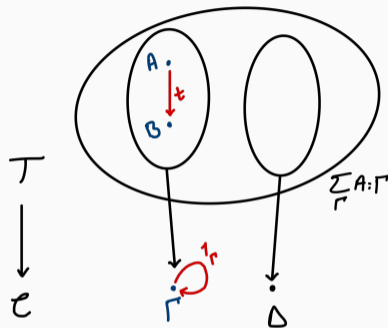
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- Term judgement $\Gamma \vdash a : A$ as a macro for $\Gamma \vdash a : \Gamma.A$ and $\Gamma \vdash \pi_A \circ a \equiv 1$

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- Judgement $\Gamma | A \tilde{\vdash} i : B$ is also a macro

CCTT: Structural Rules

Structural rules regarding the category of contexts:

$$\frac{\Gamma \text{ ctx}}{\Gamma \vdash 1_{\Gamma} : \Gamma} \text{ ctx-mor-id}$$

$$\frac{\Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta}{\Gamma \vdash s' \circ s : \Theta} \text{ ctx-mor-comp}$$

$$\frac{\Gamma \vdash s : \Delta}{\begin{array}{l} \Gamma \vdash s \circ 1_{\Gamma} \equiv s : \Delta \\ \Gamma \vdash 1_{\Delta} \circ s \equiv s : \Delta \end{array}} \text{ ctx-id-unit}$$

$$\frac{\Gamma \vdash s : \Delta \quad \Delta \vdash s' : \Theta \quad \Theta \vdash s'' : \Phi}{\Gamma \vdash s'' \circ (s' \circ s) \equiv (s'' \circ s') \circ s : \Phi} \text{ ctx-comp-assoc}$$

We have similar rules for the category of types.

See the paper for the rest of the structural rules: substitution, context extension, etc

Theorem (Soundness)

Every comprehension category models the rules of CCTT.

Next, we discuss some of the rules through the lens of subtyping.

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Coraglia and Emmenegger [CE24] observe that the vertical morphisms can be thought of as **witnesses for coercive subtyping**.

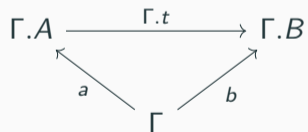
$$\Gamma \mid A \vdash t : B \quad \rightsquigarrow \quad \Gamma \vdash A \leq_t B$$

Subtyping: Subsumption

Theorem (Subsumption)

From the rules of CCTT, we can derive the following rule.

$$\frac{\Gamma \vdash A, B \text{ type} \quad \Gamma \vdash A \leq_t B \quad \Gamma \vdash a : A}{\Gamma \vdash \Gamma.t \circ a : B}$$



$\Gamma.t$ is like a **coercion function** for $A \leq_t B$.

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We also have rules for weakening and substitution of the coercion witnesses.

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Type Formers Are Functors

- Type formers make new types from old
- ↪ Type formers should also act on **morphisms of types**

Steps Towards Type Formers

1. Extend CCTT with a type former (e.g. Σ -types) and show soundness.
2. Extend CCTT with subtyping for the type former and show soundness

We look at subtyping for Σ -types to keep things simple.

Rules for Subtyping for Σ -types

1. We want to have the following rule:

$$\frac{\begin{array}{l} \Gamma \vdash A, A' \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma.A' \vdash B' \text{ type} \\ \Gamma \vdash A \leq_f A' \quad \Gamma.A \vdash B \leq_g B'[\Gamma.f] \end{array}}{\Gamma \vdash \Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'}$$

Σ acts covariantly on both arguments.

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2. The coercion function for $\Sigma_A B \leq_{\Sigma(f,g)} \Sigma_{A'} B'$ should act as follows:

$$\Gamma.\Sigma_A B \xrightarrow{\text{proj}_{\Sigma_A B}} \Gamma.A.B \xrightarrow{\chi_0 g} \Gamma.A.B'[\chi_0 f] \xrightarrow{\chi_0 f.B'} \Gamma.A'.B' \xrightarrow{\text{pair}_{\Sigma_{A'} B'}} \Gamma.\Sigma_{A'} B'$$

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3. Rules for functoriality for $\Sigma(-, -)$

Semantic Structure for Subtyping for Σ -Types

Definition

$(\mathcal{C}, \mathcal{T}, \rho, \chi)$ has subtyping for Σ -types if it has

1. dependent sums and
2. for each $f : A \rightarrow A'$ in \mathcal{T}_Γ and $g : B \rightarrow B'[\chi_0 f]$ in $\mathcal{T}_{\Gamma.A}$, a morphism in \mathcal{T}_Γ

$$\Sigma_f g : \Sigma_A B \rightarrow \Sigma_{A'} B'$$

3. $\chi_0(\Sigma_f g)$ is the following composite

$$\Gamma.\Sigma_A B \xrightarrow{\text{proj}_{\Sigma_A B}} \Gamma.A.B \xrightarrow{\chi_0 g} \Gamma.A'.B'[\chi_0 f] \xrightarrow{\chi_0 f.B'} \Gamma.A'.B' \xrightarrow{\text{pair}_{\Sigma_{A'} B'}} \Gamma.\Sigma_{A'} B'$$

4. $\Sigma_{(-)}(-)$ preserves identities and composition

Theorem

Any comprehension category with subtyping for Σ -types models CCTT extended with subtyping for Σ -types.

Sanity Check

When χ is fully faithful,

- our Σ -structures are equivalent to Lumsdaine and Warren's [LW15]
- Jacobs' structure for Σ -types [Jac93] gives rise to ours

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“From Semantics to Syntax: A Type Theory For Comprehension Categories”

- CCTT reflects the structure of a comprehension category.
- Gain back the ‘extra dimension’ of type morphisms; captures coercive subtyping.

Current Work

- AWFSs give rise to comprehension categories
- We aim to characterise those comprehension categories (with type constructions) that arise from AWFSs

Link between AWFSs and Comprehension Categories

Our Goal

- Characterise comprehension categories that arise from AWFSs
- Type theory for AWFS; reflect interesting features of AWFS to syntax

Intermediate Goals

- Understand better the link between comprehension categories and AWFSs
- An equivalence of categories:

$$\text{AWFS}^{\text{lax}} + \text{some structure} \xleftrightarrow{\simeq} \text{CompCat}^{\text{strict}} + \text{some structure}$$

Thank you for your attention!

References

- [ALN24] Benedikt Ahrens, Peter LeFanu Lumsdaine, and Paige Randall North. “Comparing Semantic Frameworks for Dependently-Sorted Algebraic Theories”. In: *Programming Languages and Systems: 22nd Asian Symposium, APLAS 2024, Kyoto, Japan, October 22-24, 2024, Proceedings*. Kyoto, Japan: Springer-Verlag, 2024, pp. 3–22. ISBN: 978-981-97-8942-9. DOI: 10.1007/978-981-97-8943-6_1. URL: https://doi.org/10.1007/978-981-97-8943-6_1.
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