

# Concurrent Strategies as Street Fibrations

Hugo Paquet, Inria & ENS, Paris

Glynn Winskel, Queen Mary University of London

# Concurrent strategies

A mathematical theory of games and strategies based on event structures.

*[Winskel–Rideau 2011, also Melliès, Faggian–Piccolo, Hirschowitz–Eberhart, Clairambault]*

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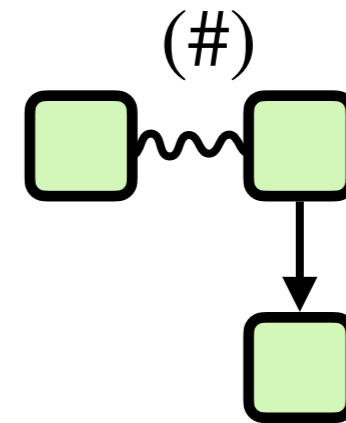
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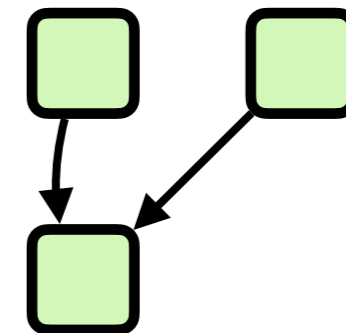
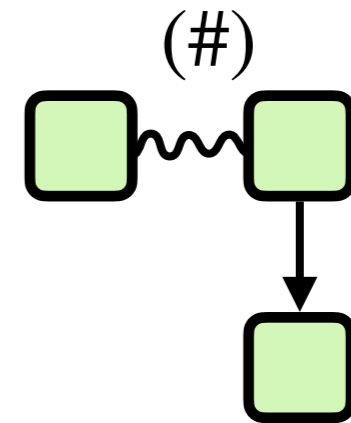
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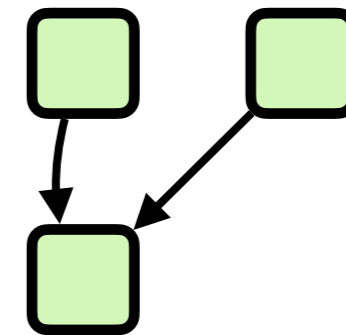
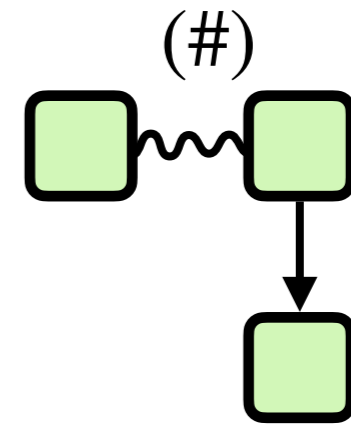
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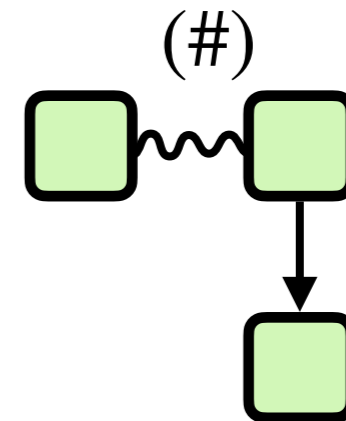
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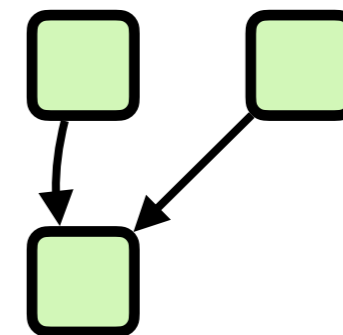
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**Def.** A morphism  $(E, \leq, \#) \rightarrow (E', \leq', \#')$  is a function  $f : E \rightarrow E'$  which preserves configurations and is locally injective.

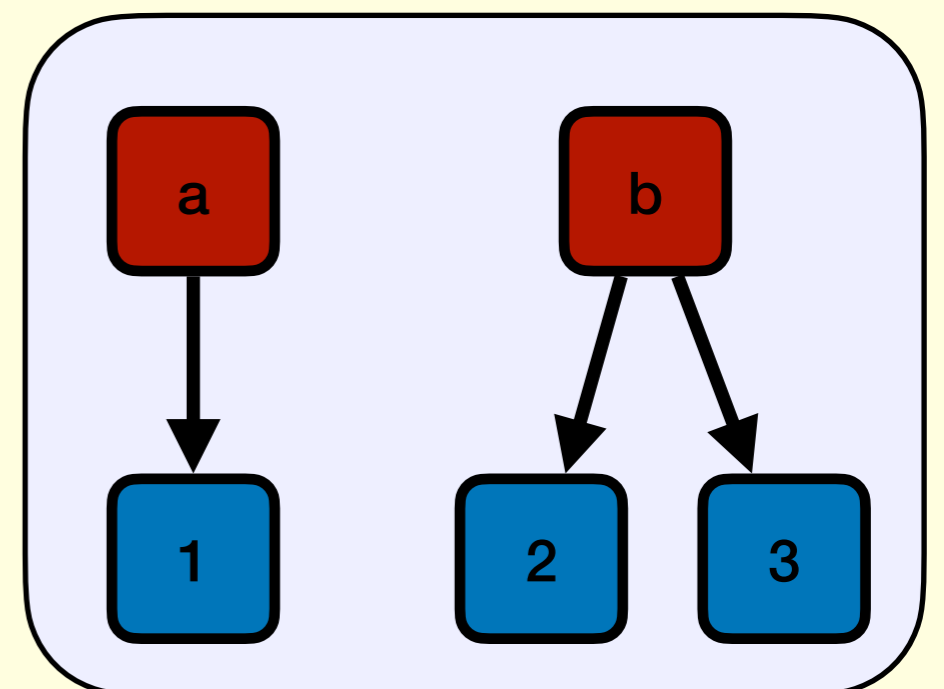
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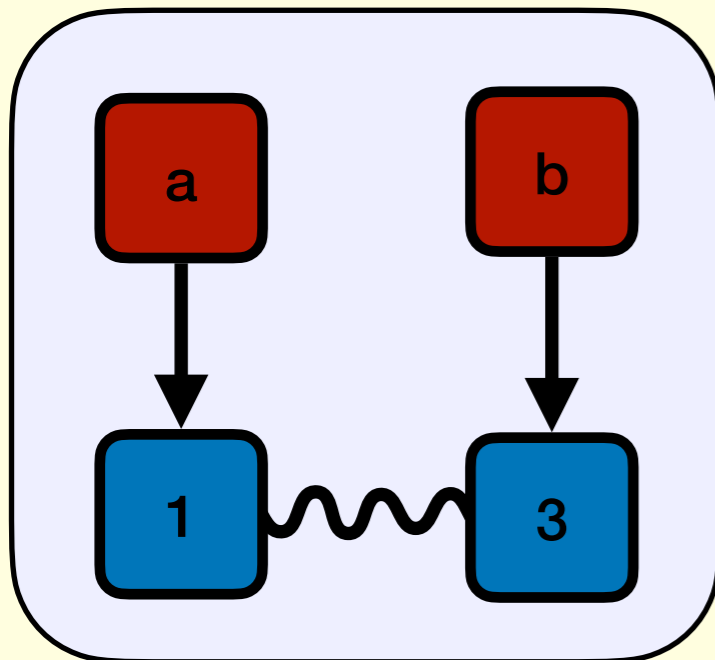
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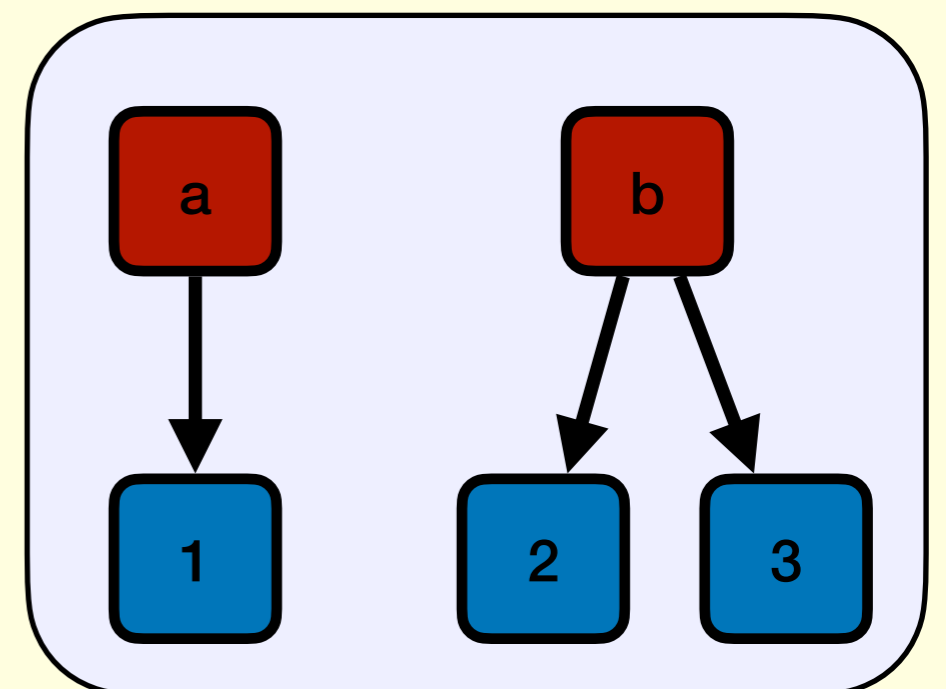
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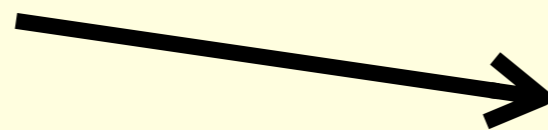
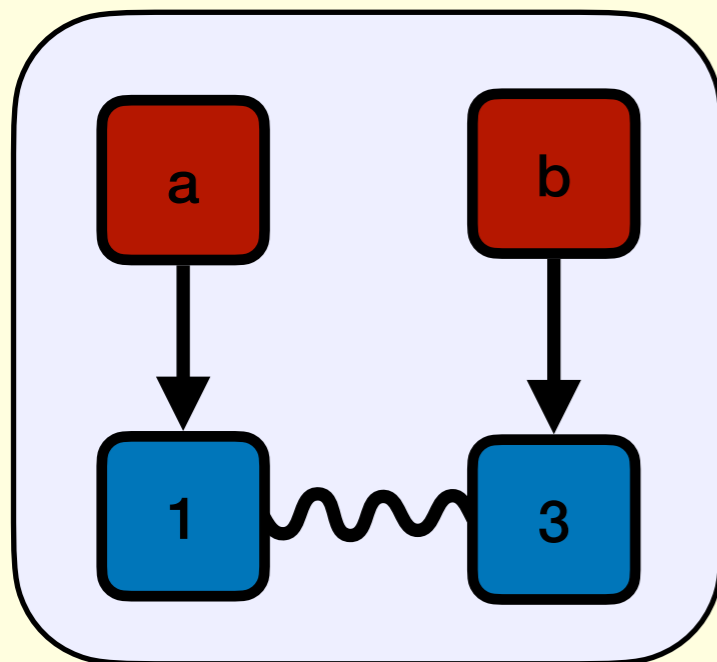
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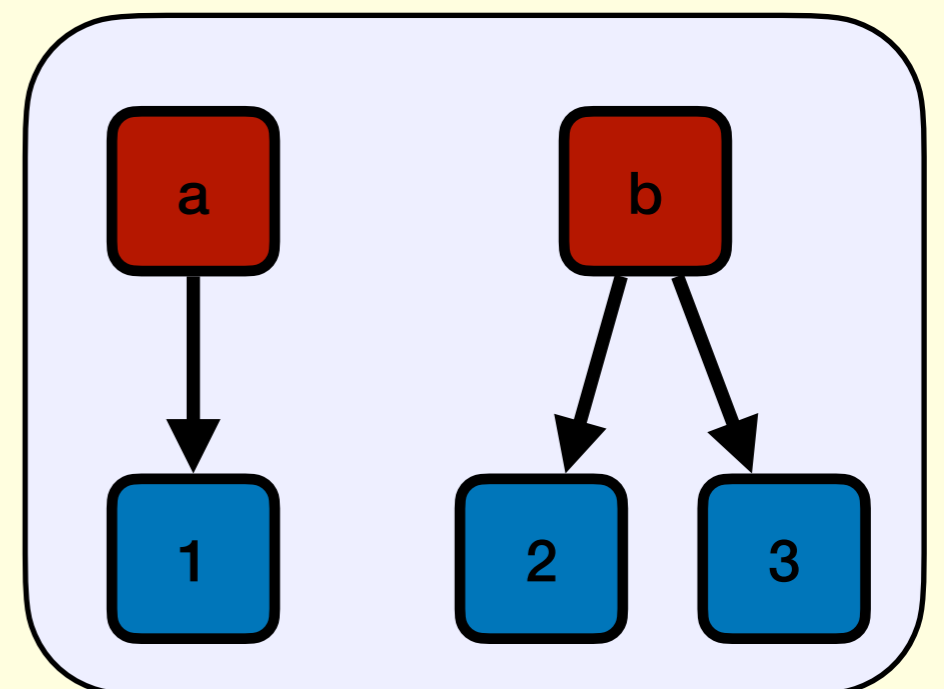
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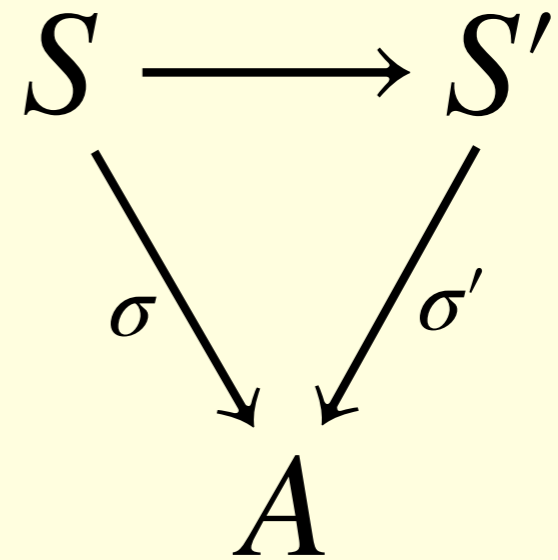
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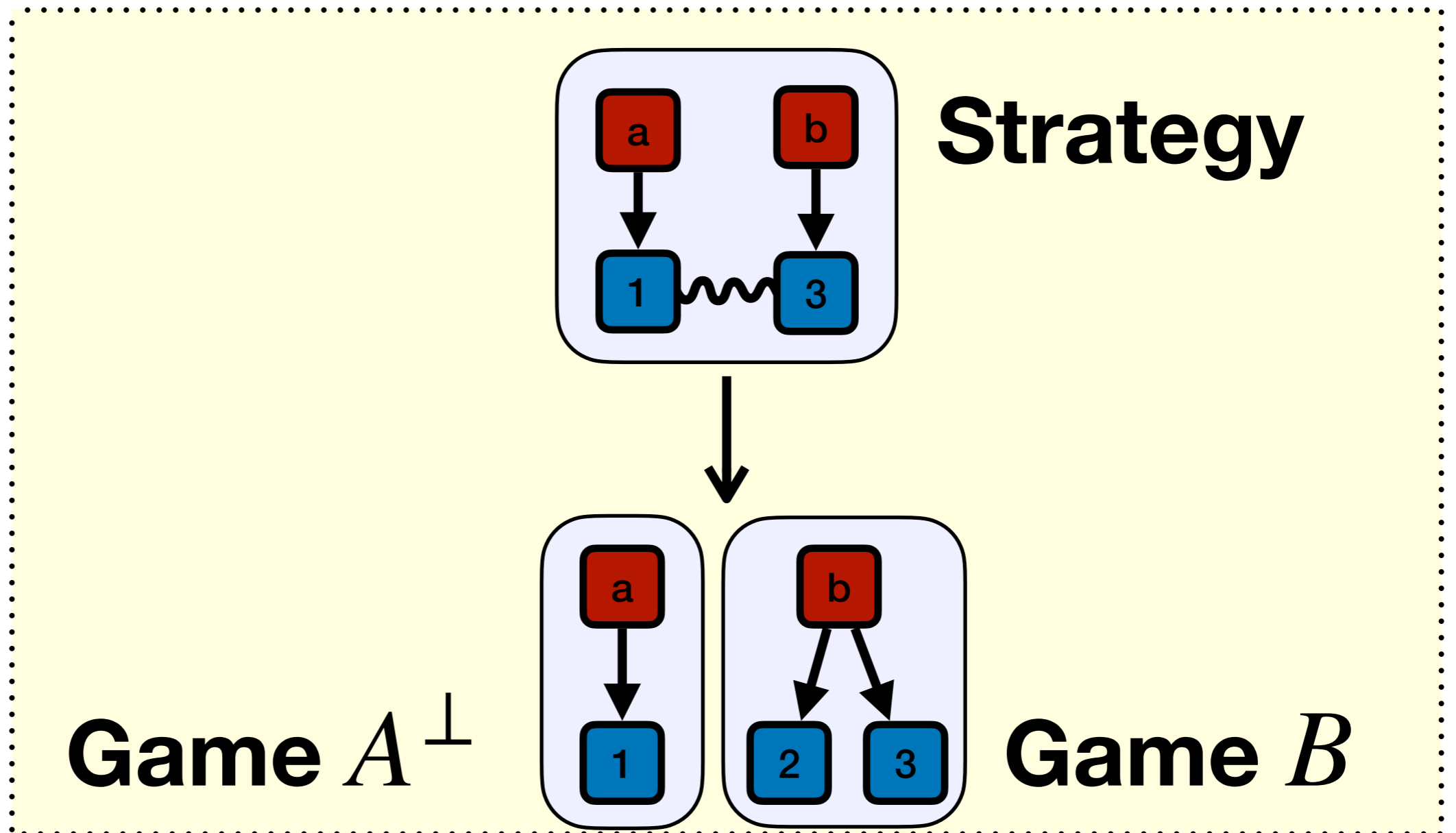
*Morphisms:*



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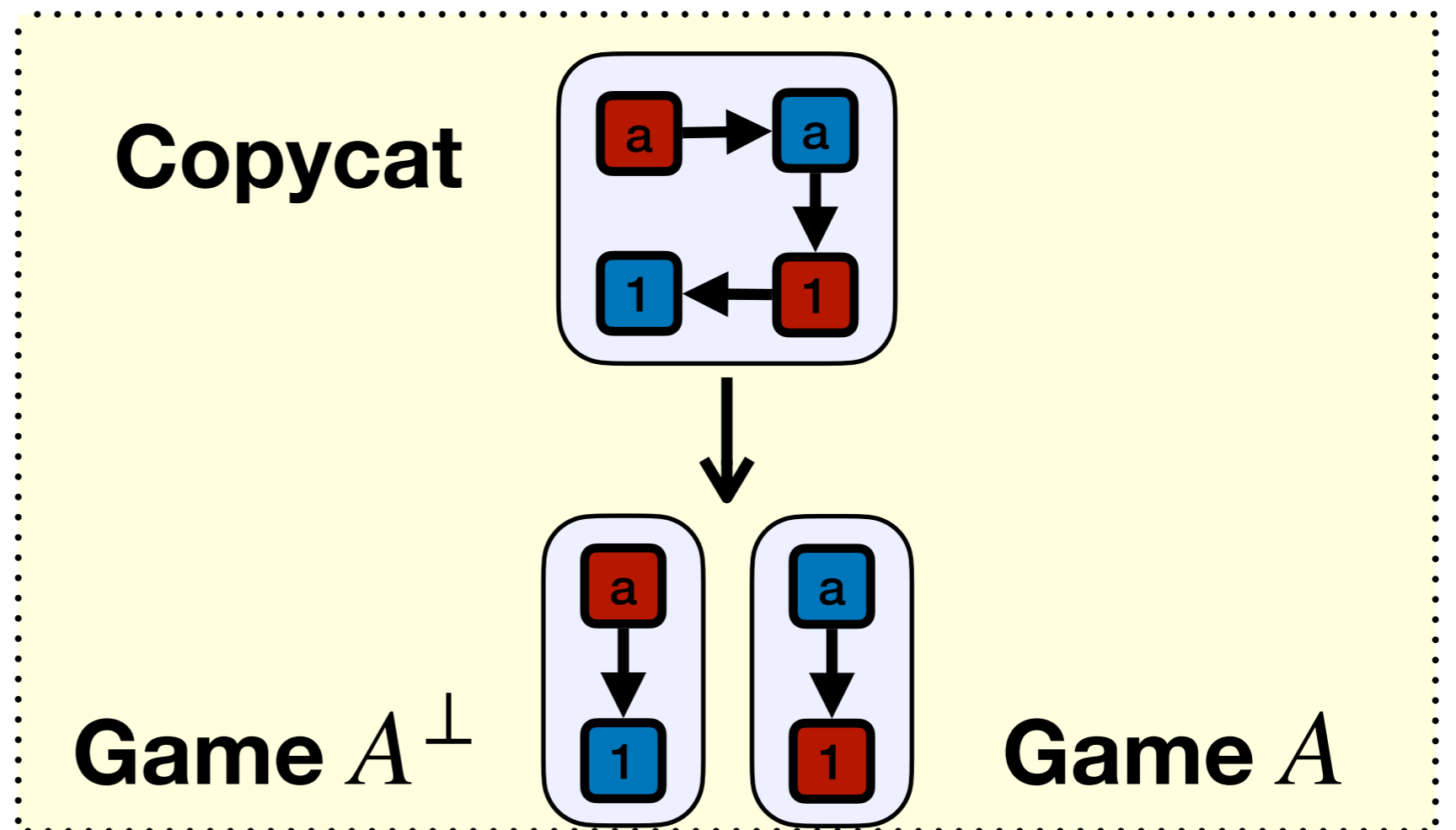
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[Winskel 2011-13]

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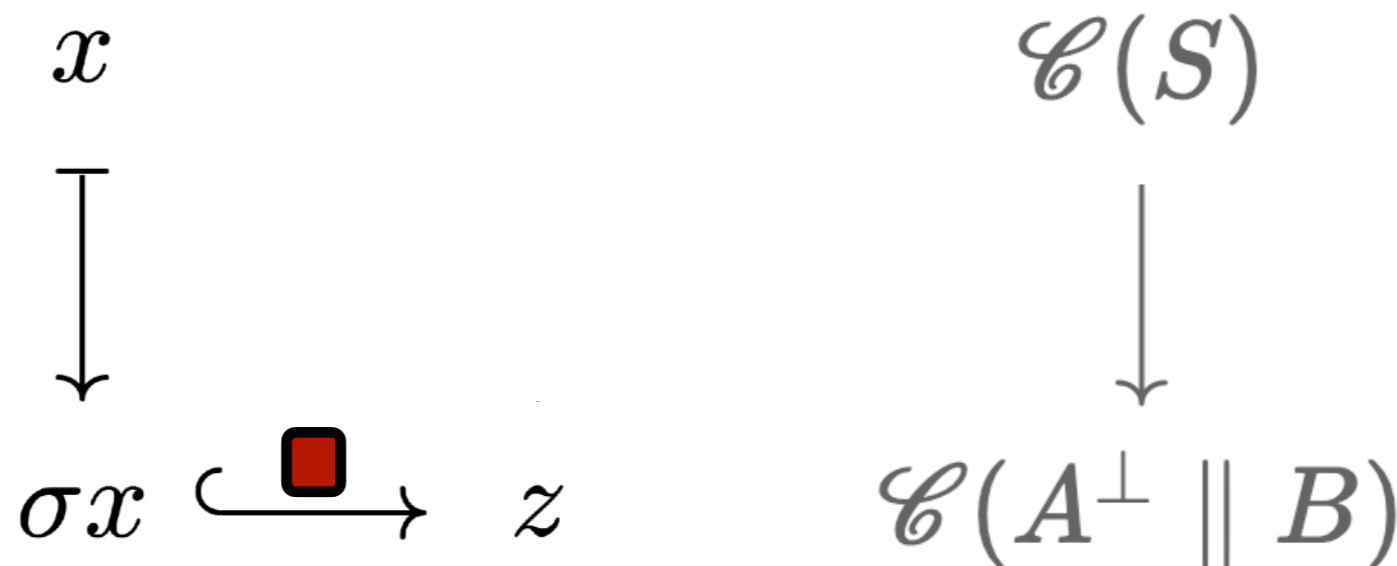
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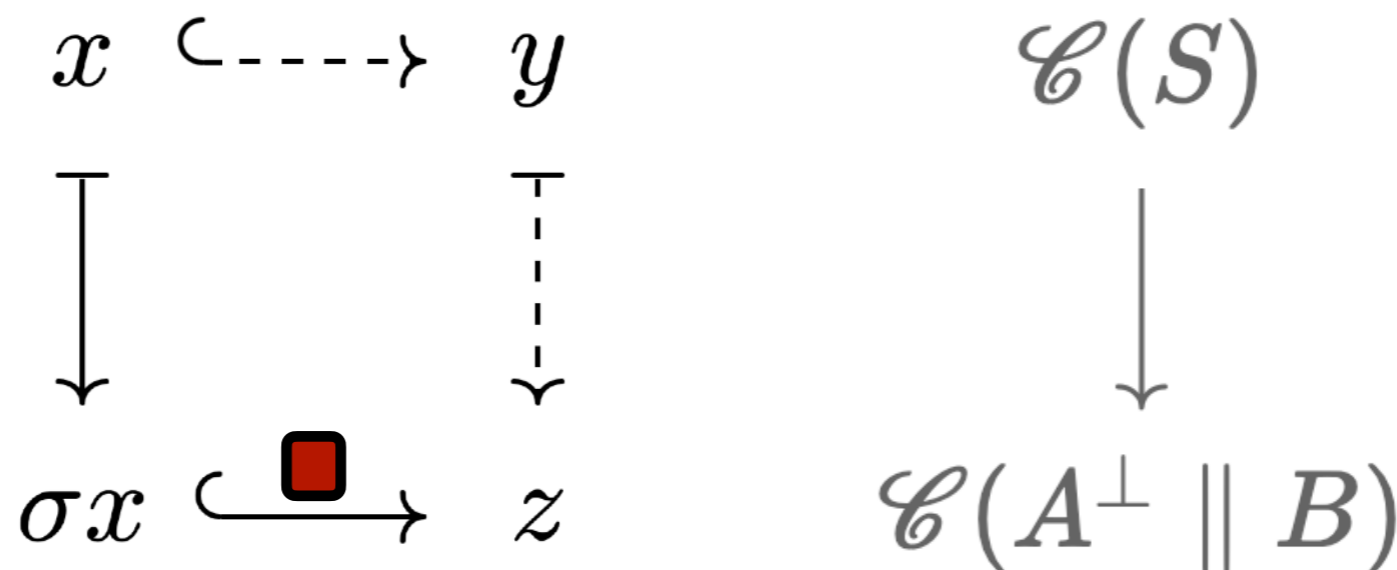
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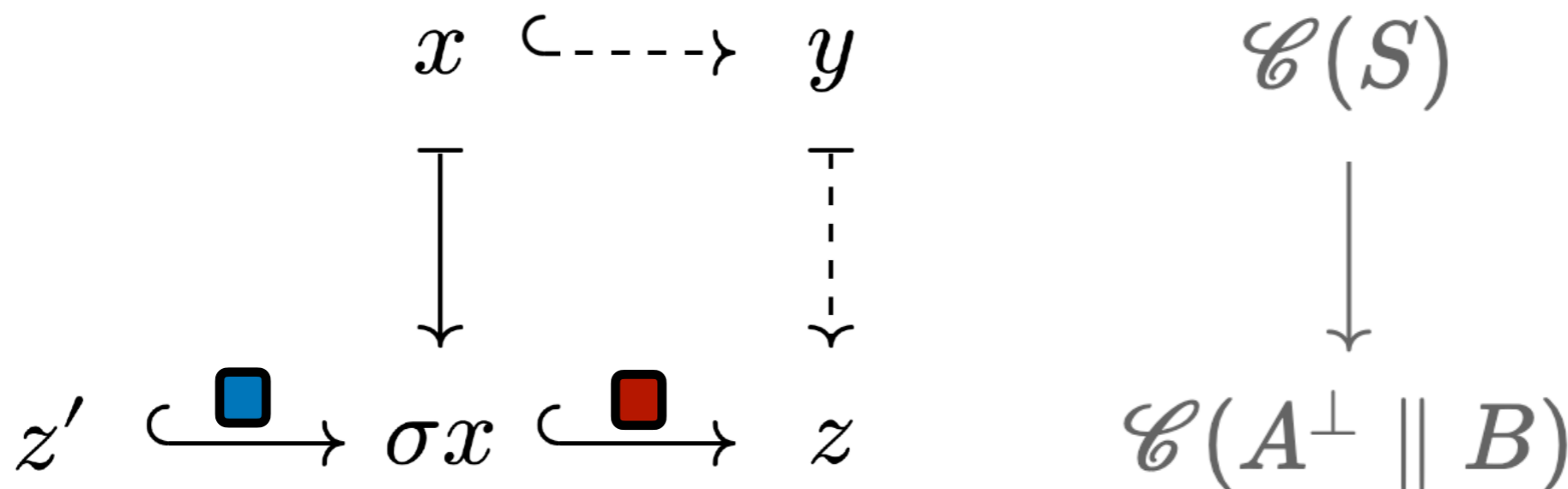
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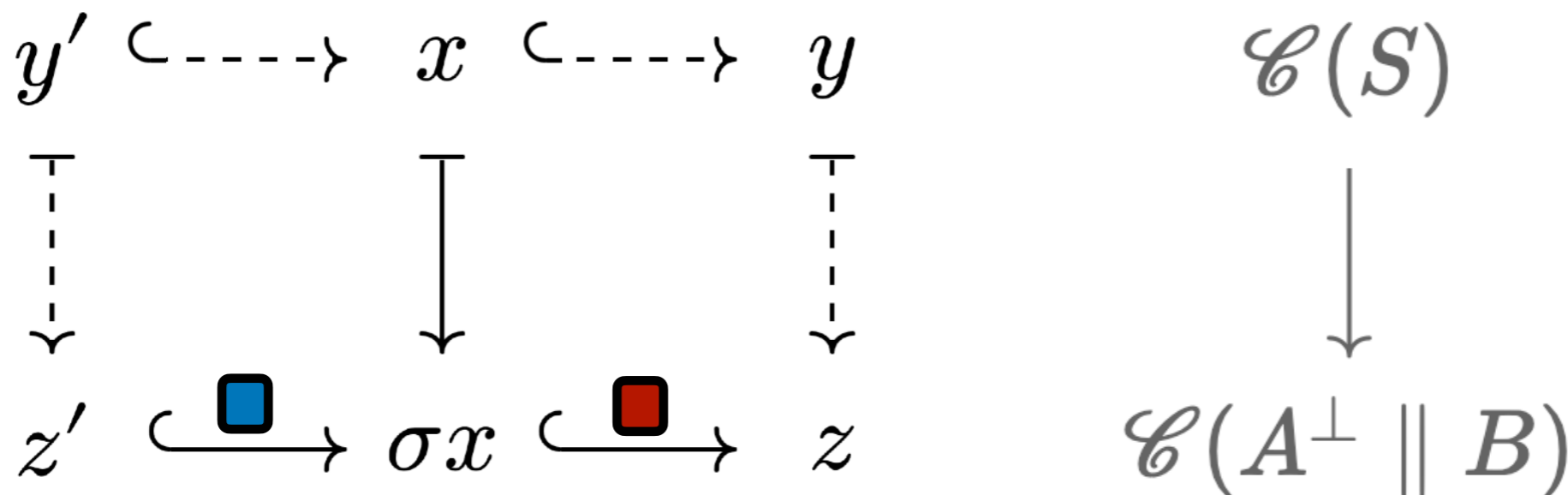
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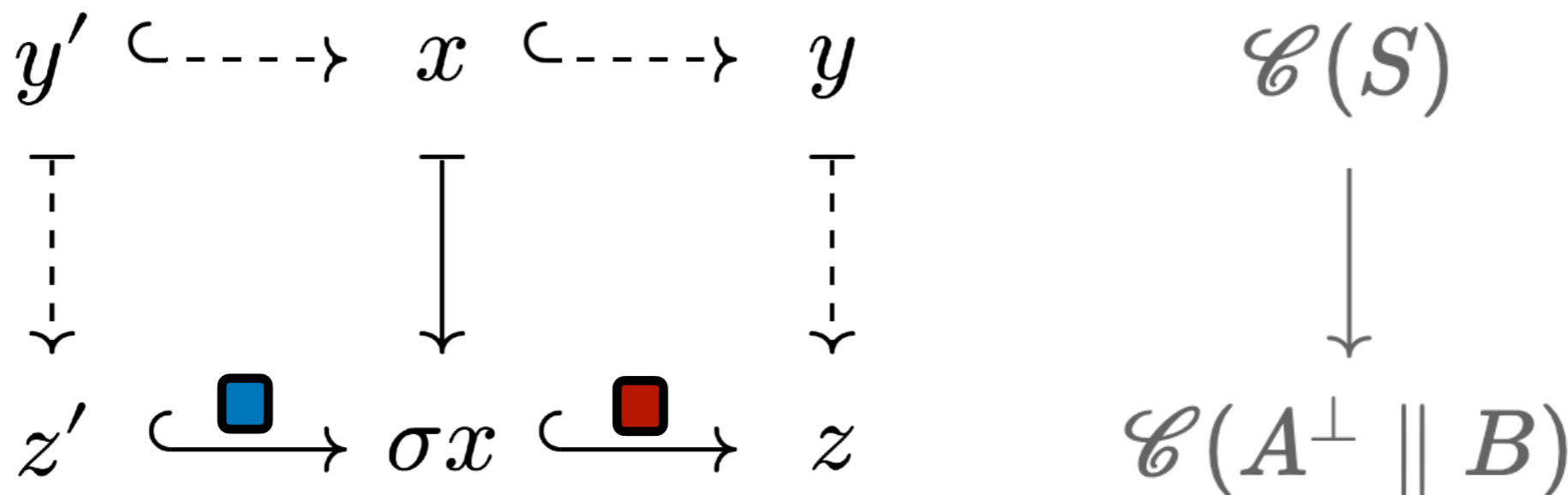
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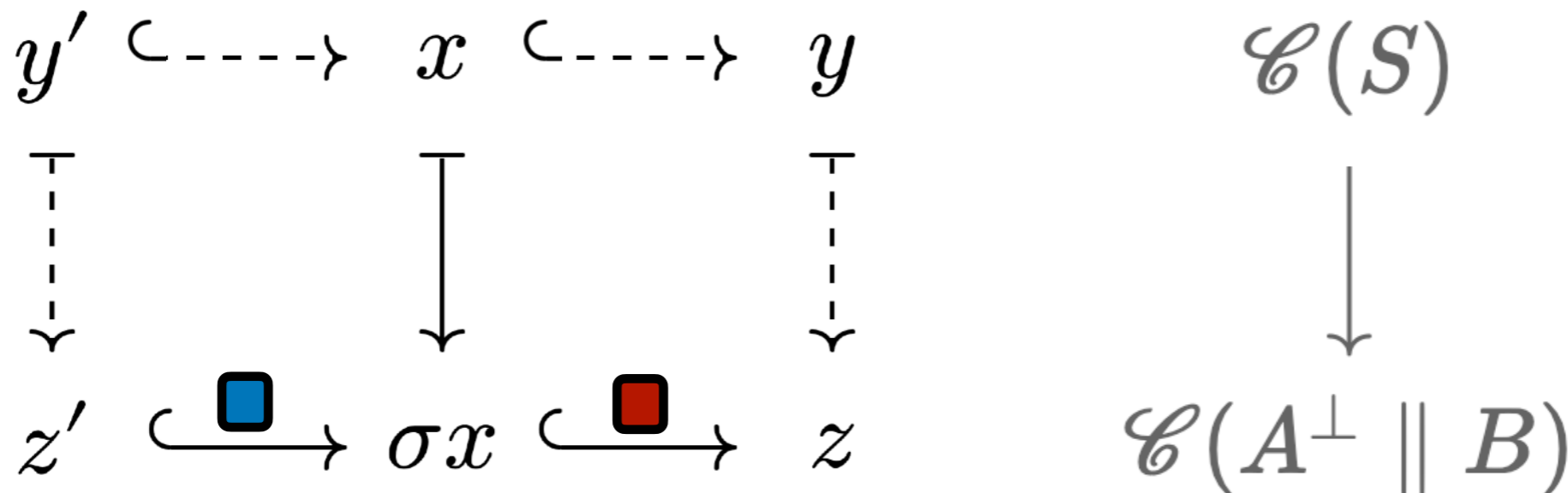
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(the "Scott order")



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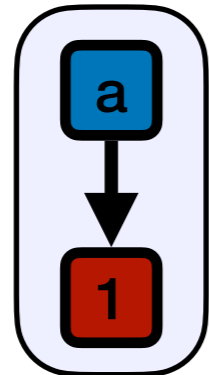
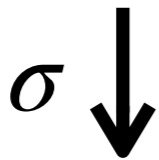
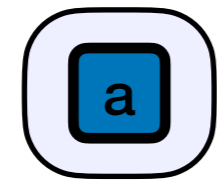
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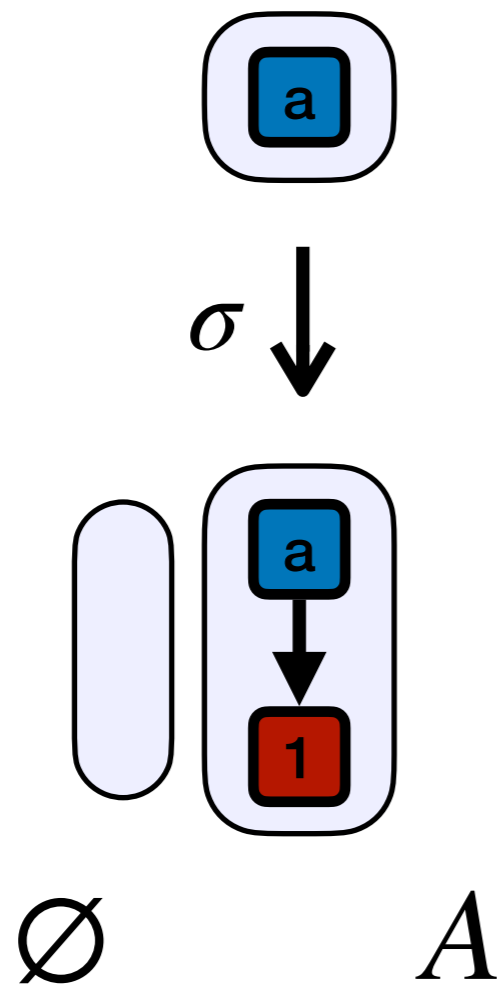
This paper: **games with symmetry and strategies as Street fibrations.**

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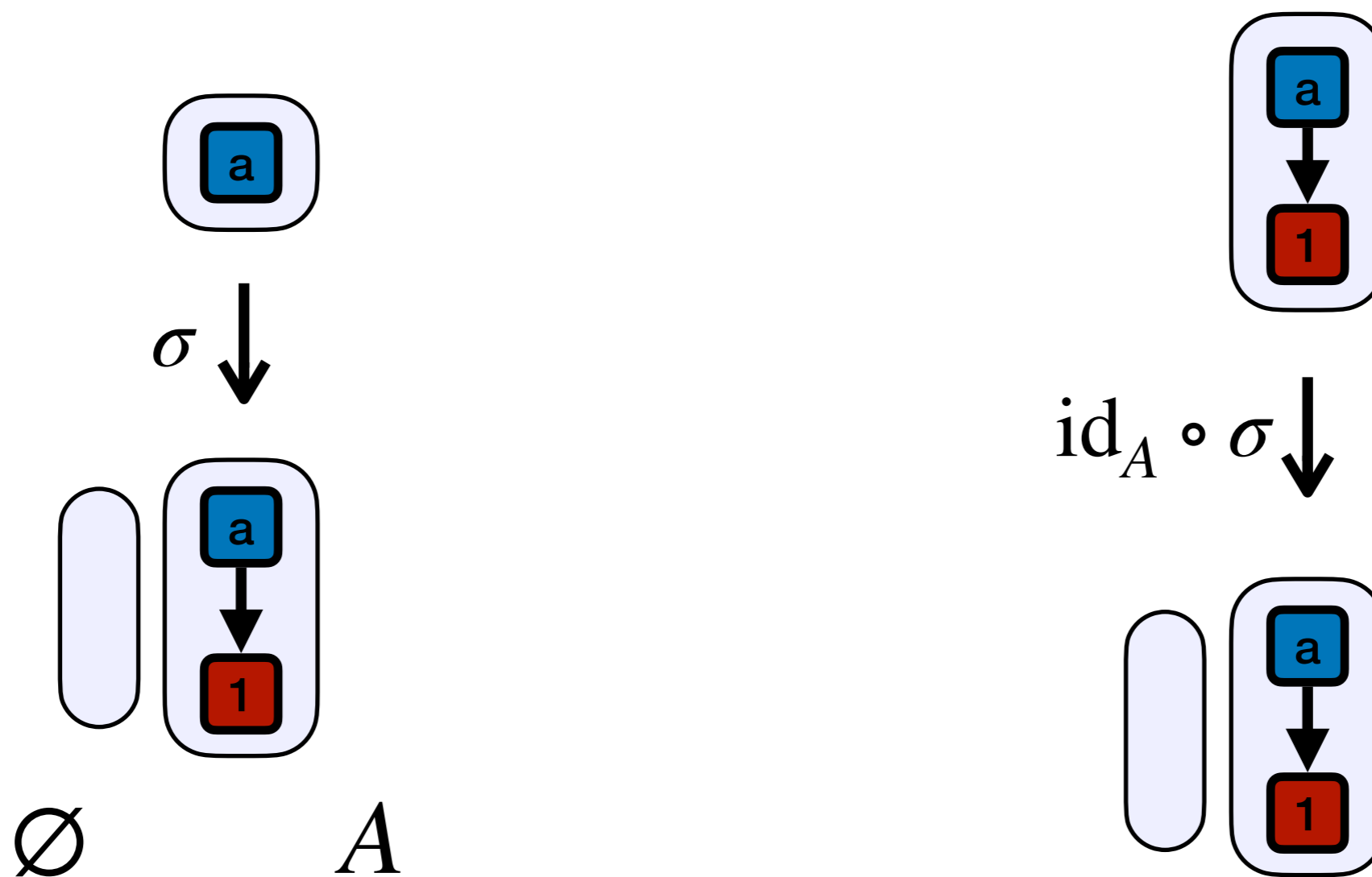


*A*

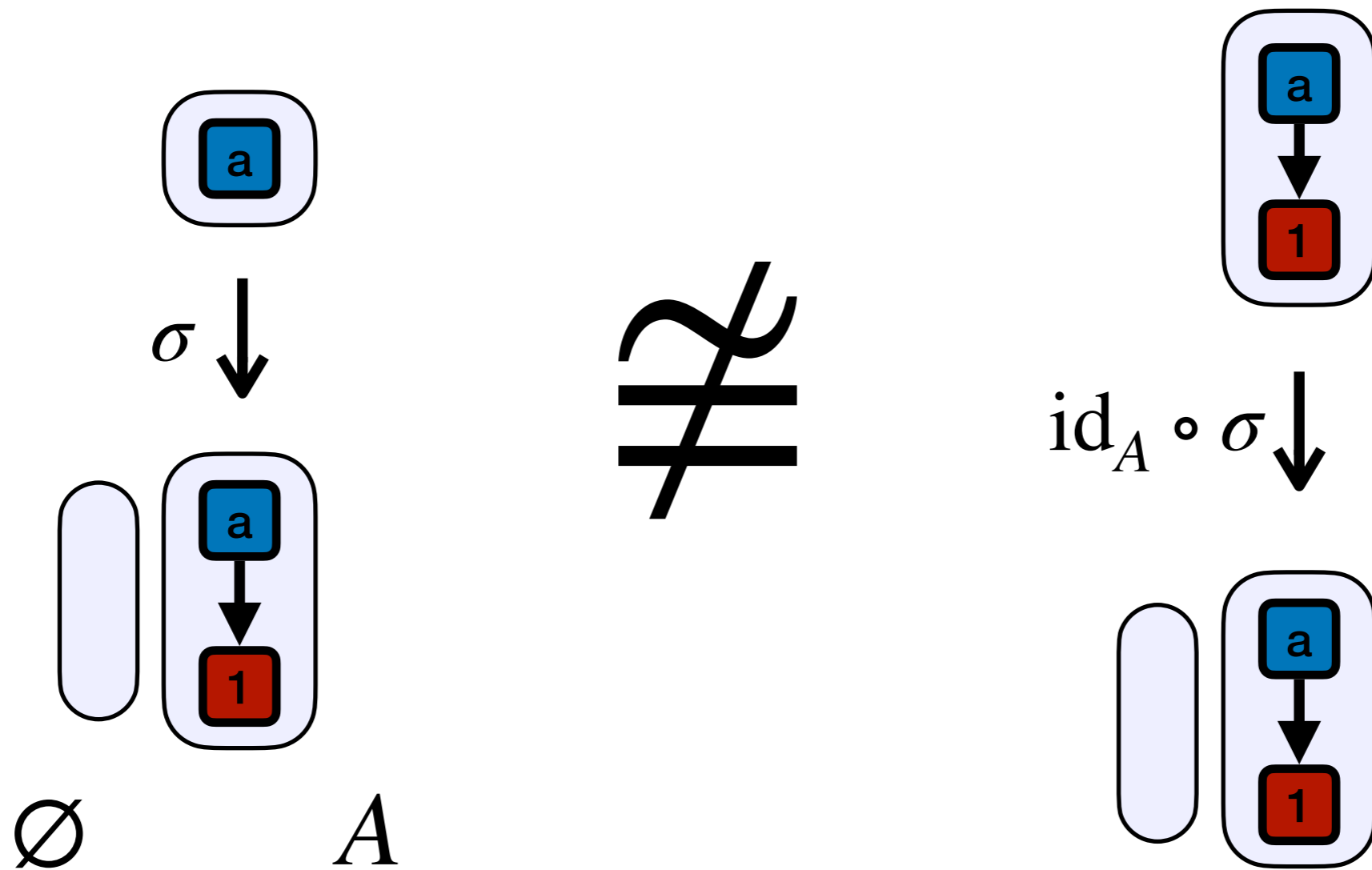
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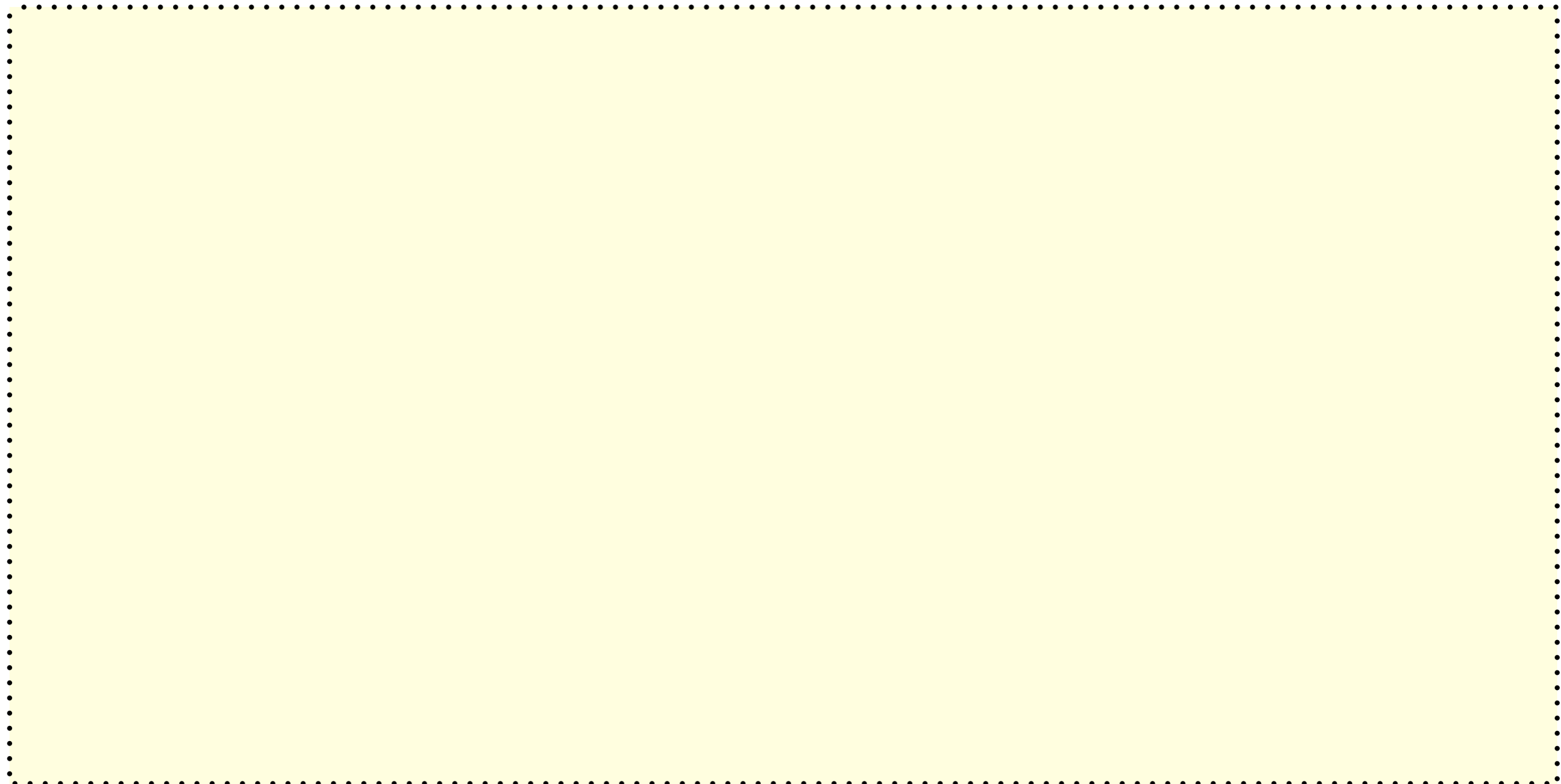
# Rest of the talk.

- I. Games and strategies with symmetry
- II. Strategies as Street fibrations
- III. Other results

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*[Winskel 2007]*

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# Games and strategies with symmetry



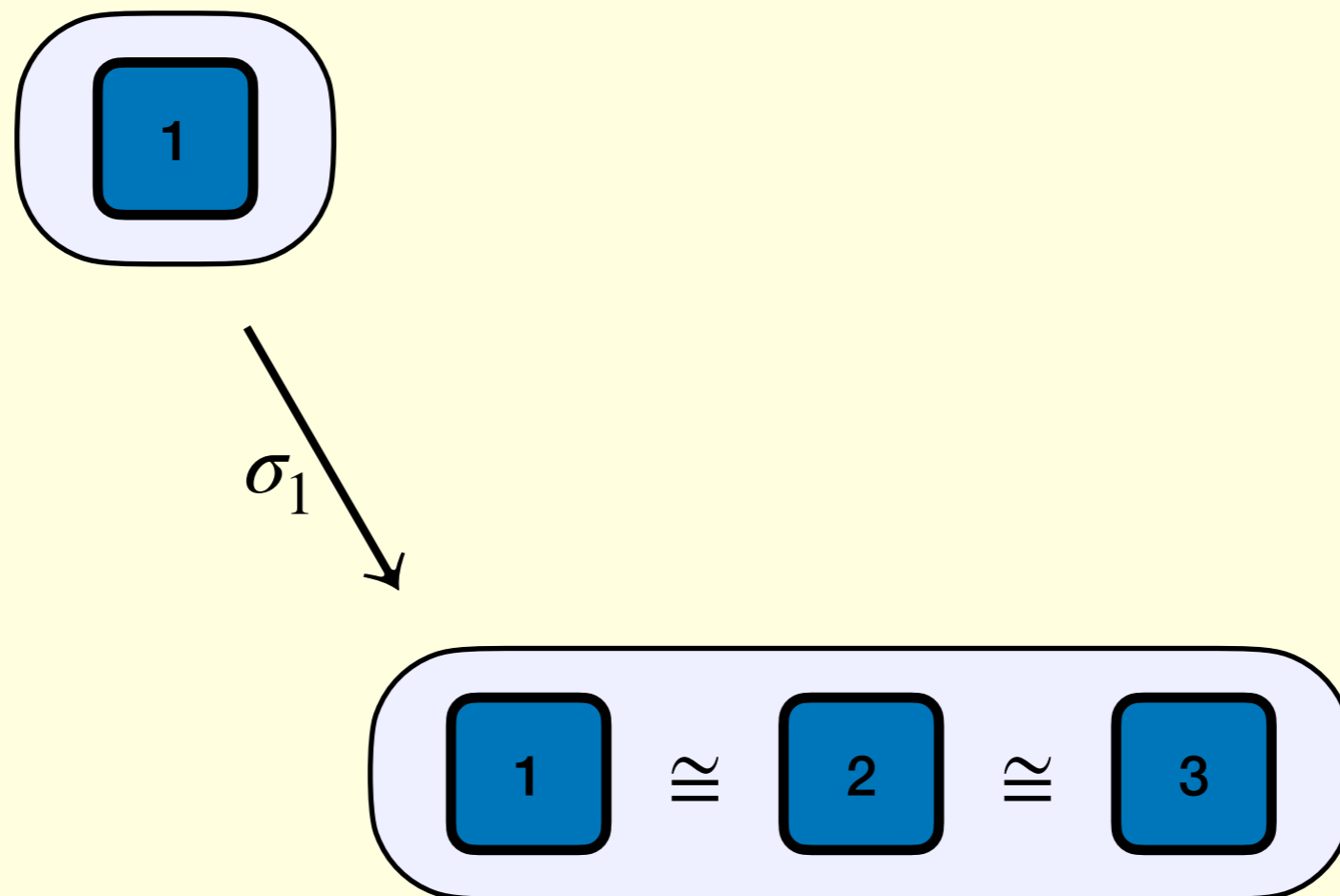
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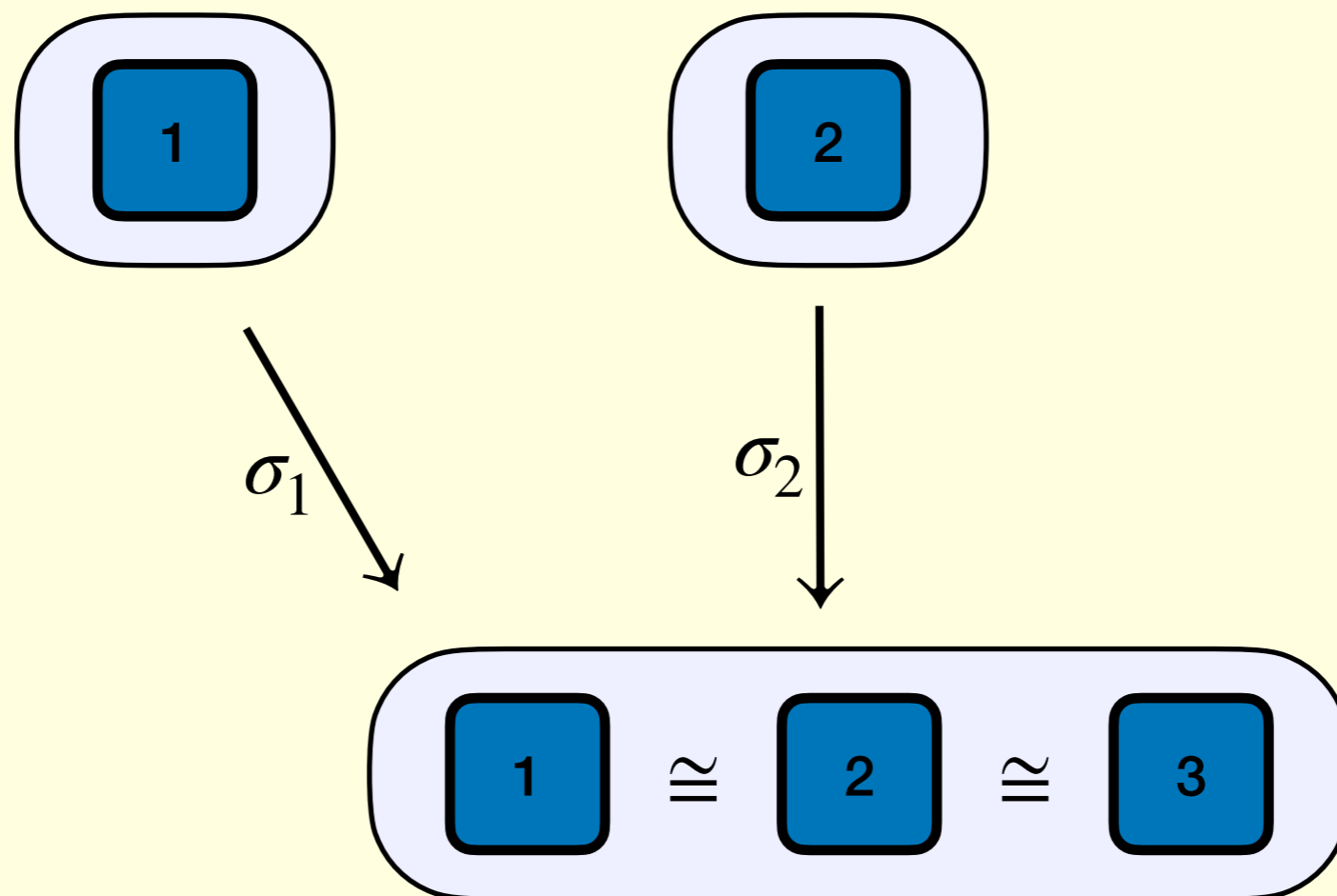
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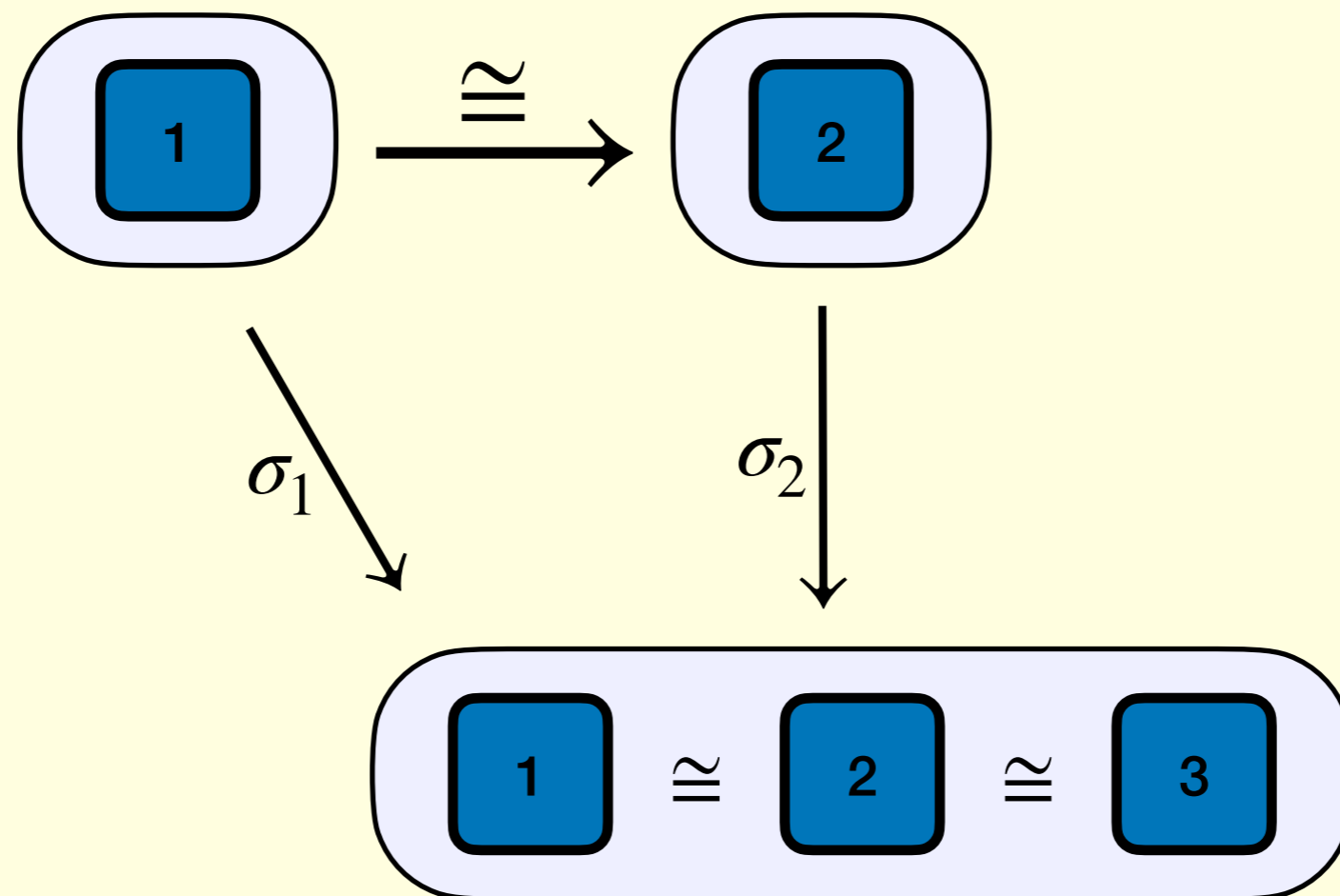
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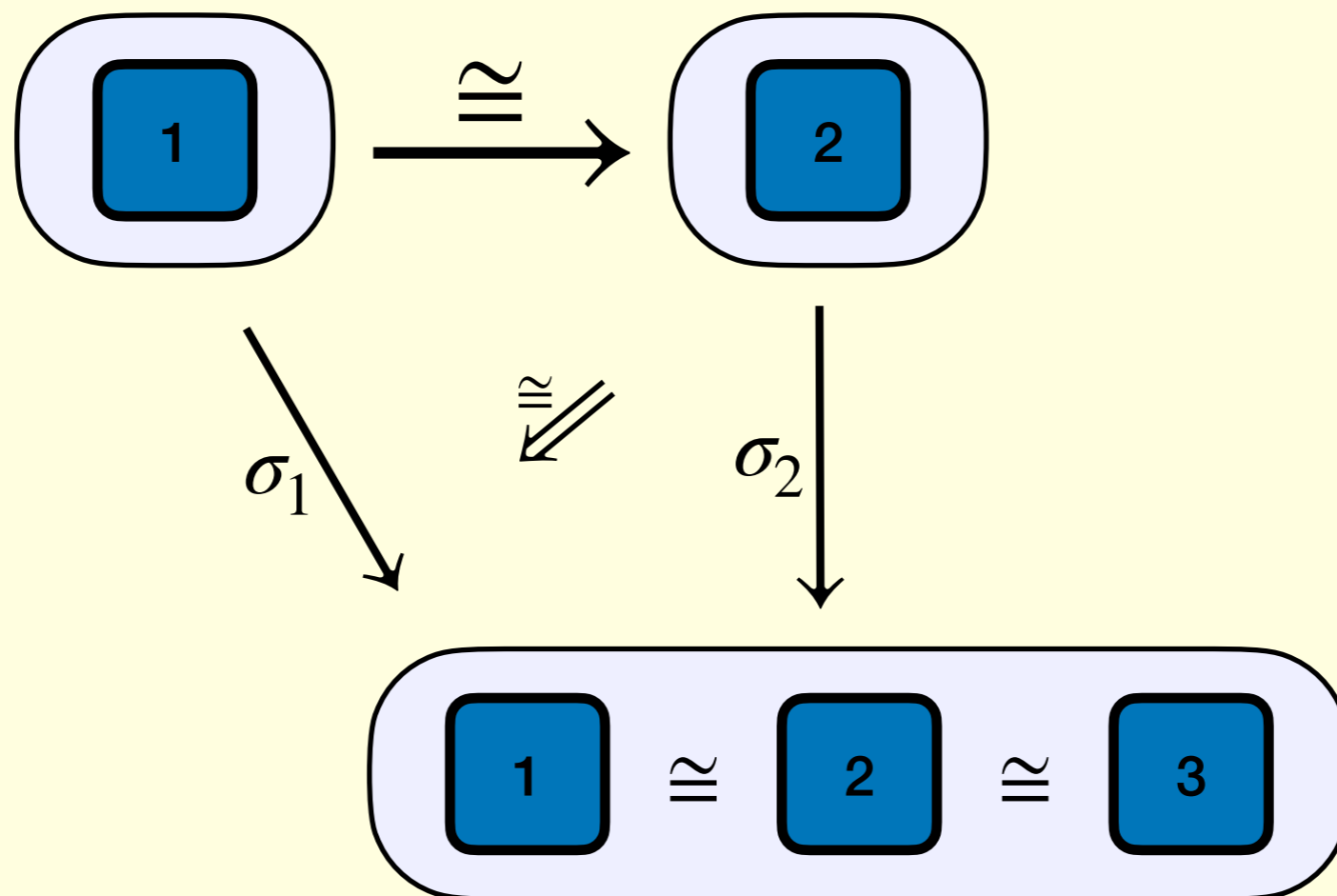
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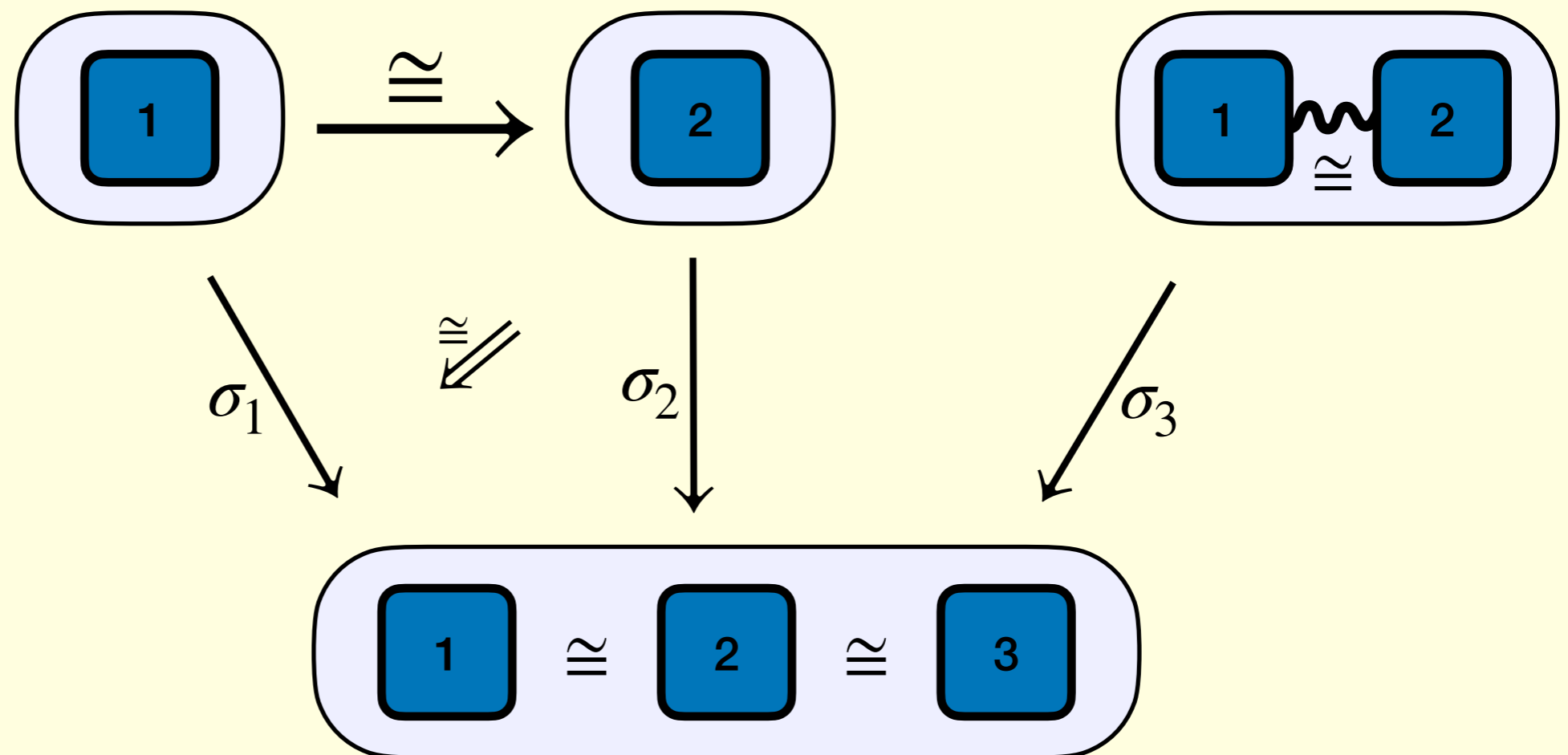
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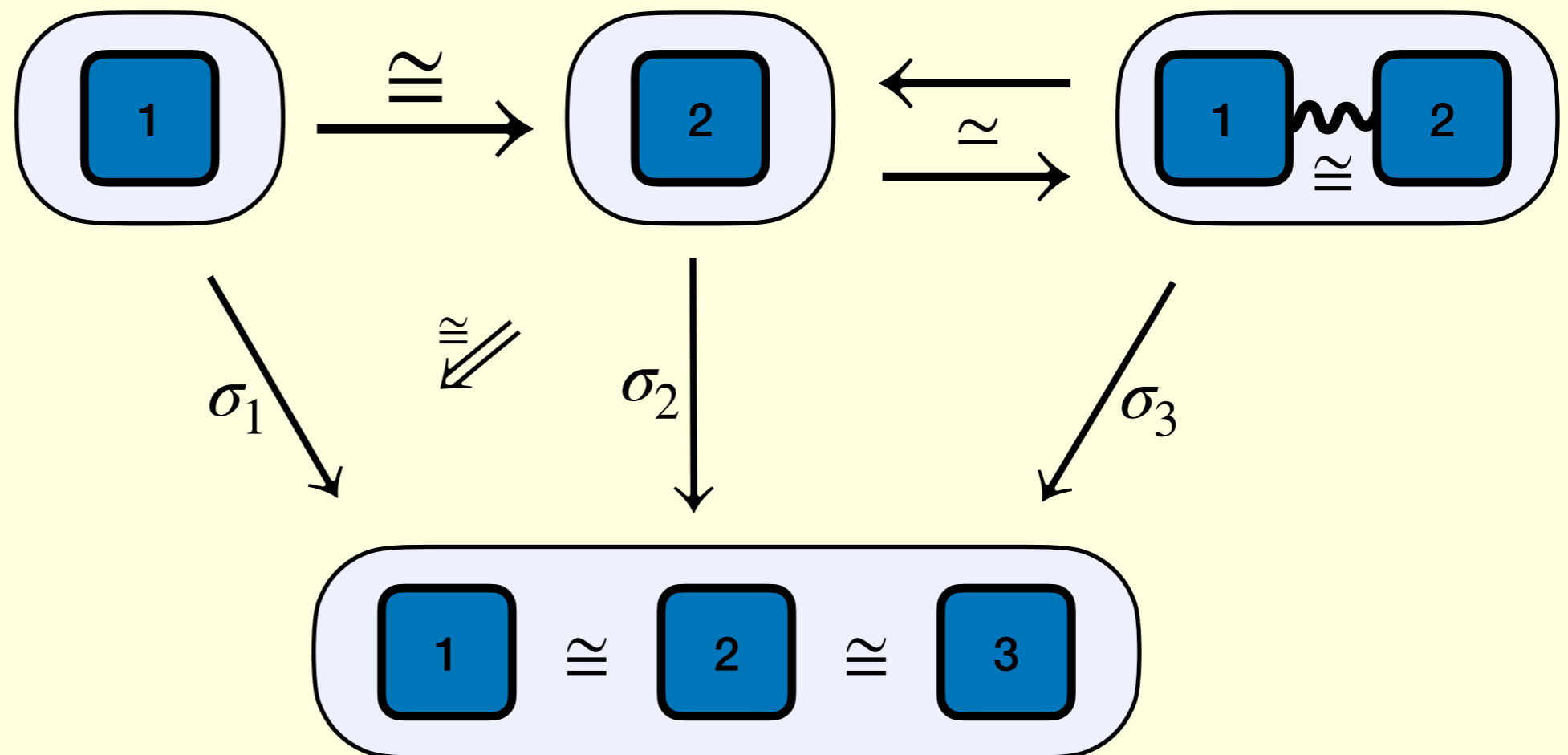
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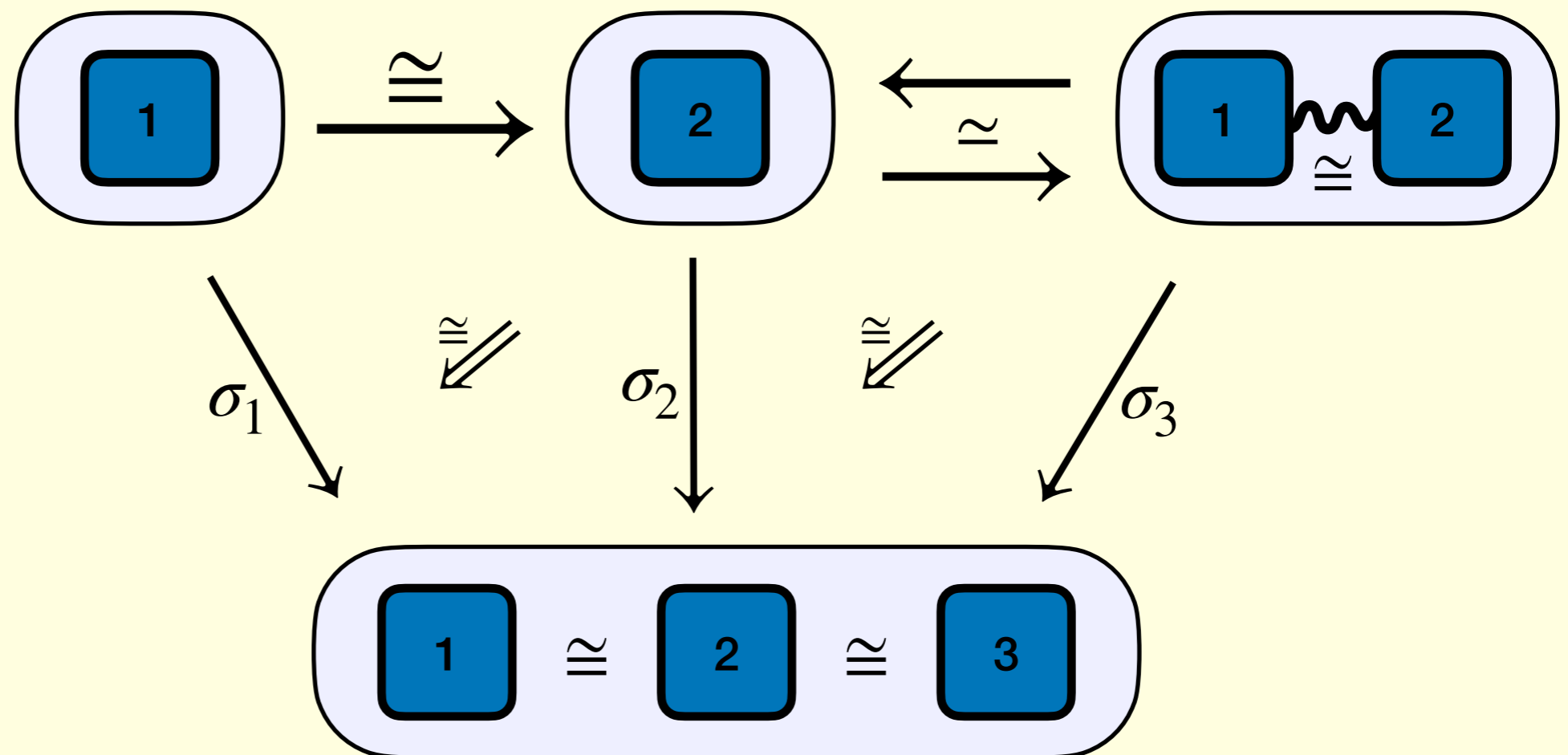
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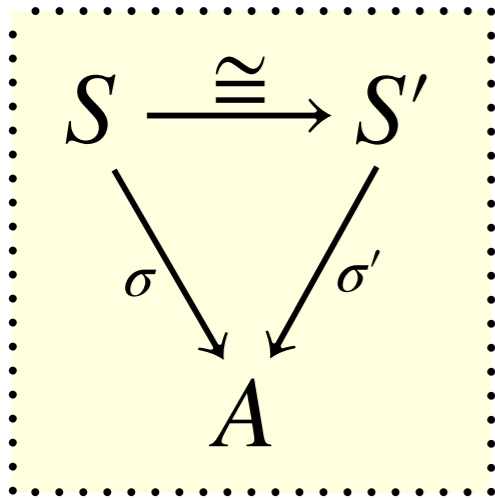
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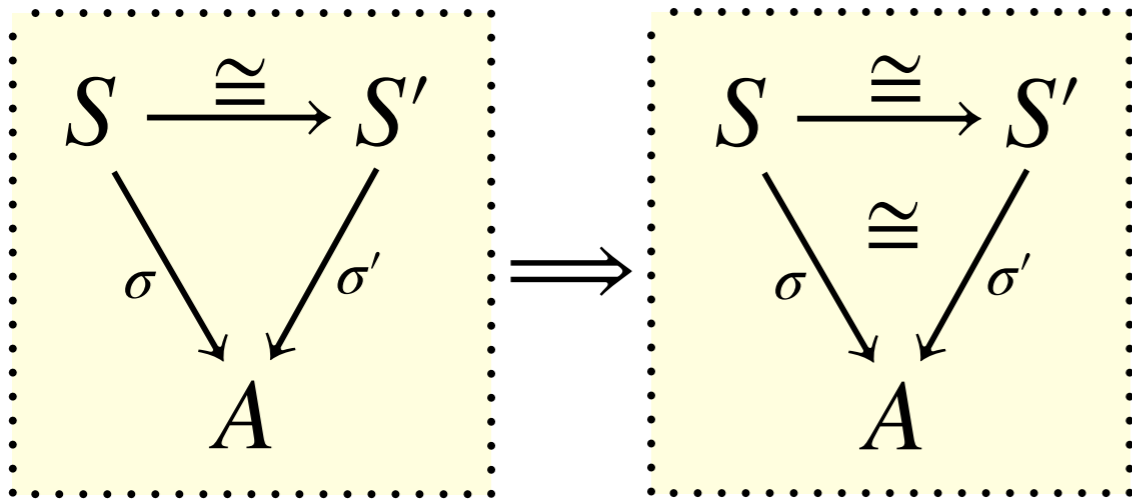


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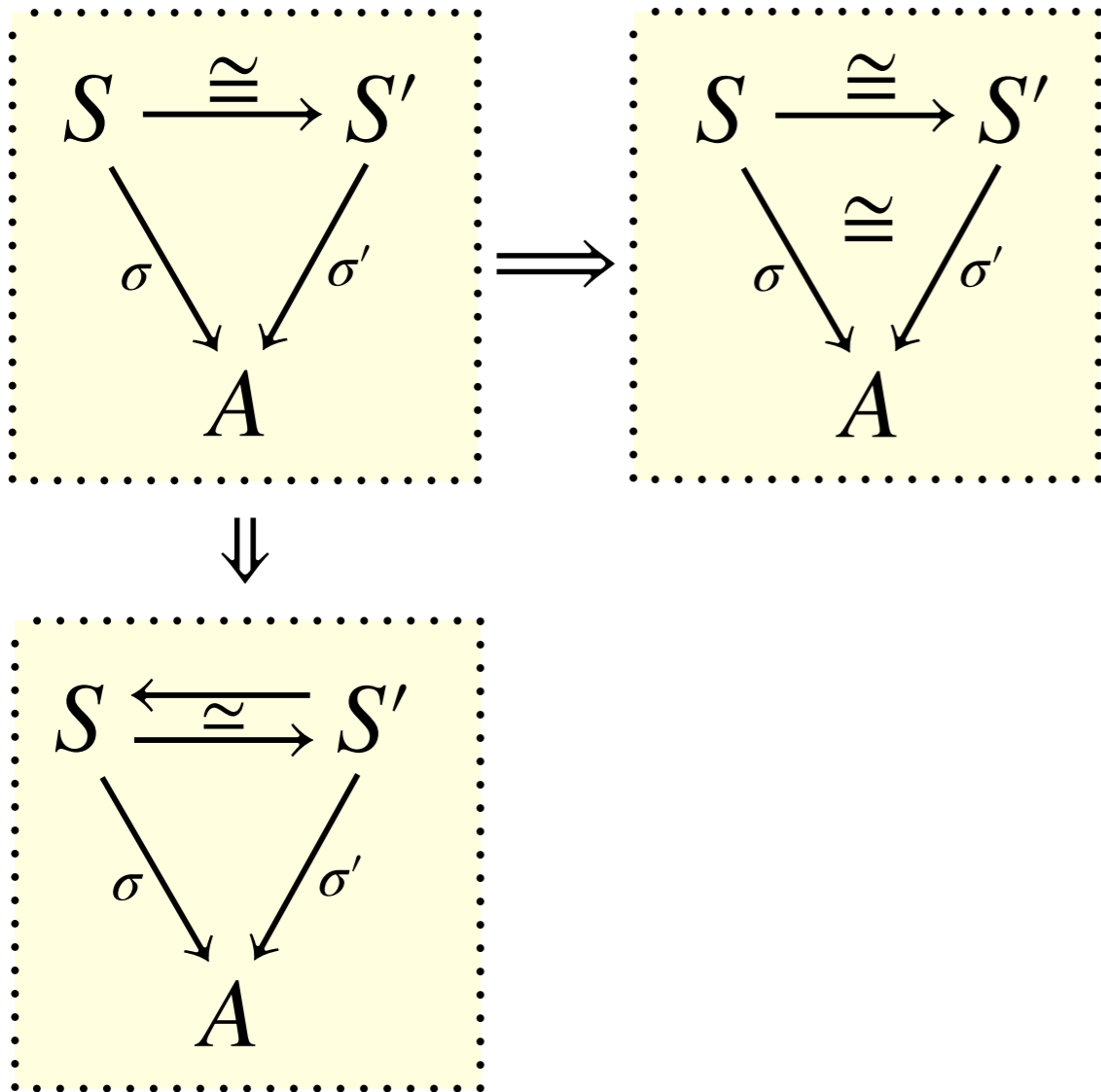
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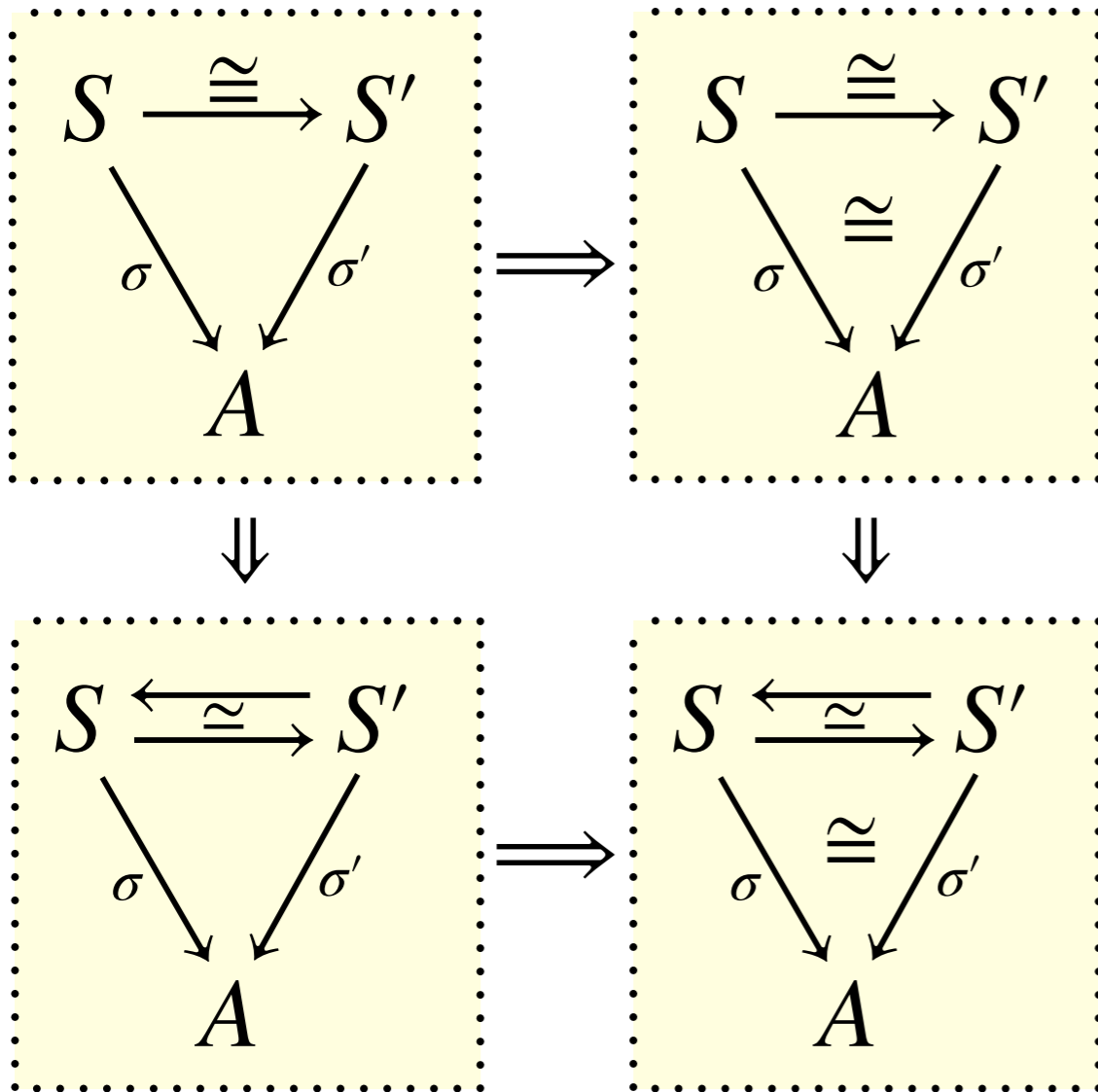
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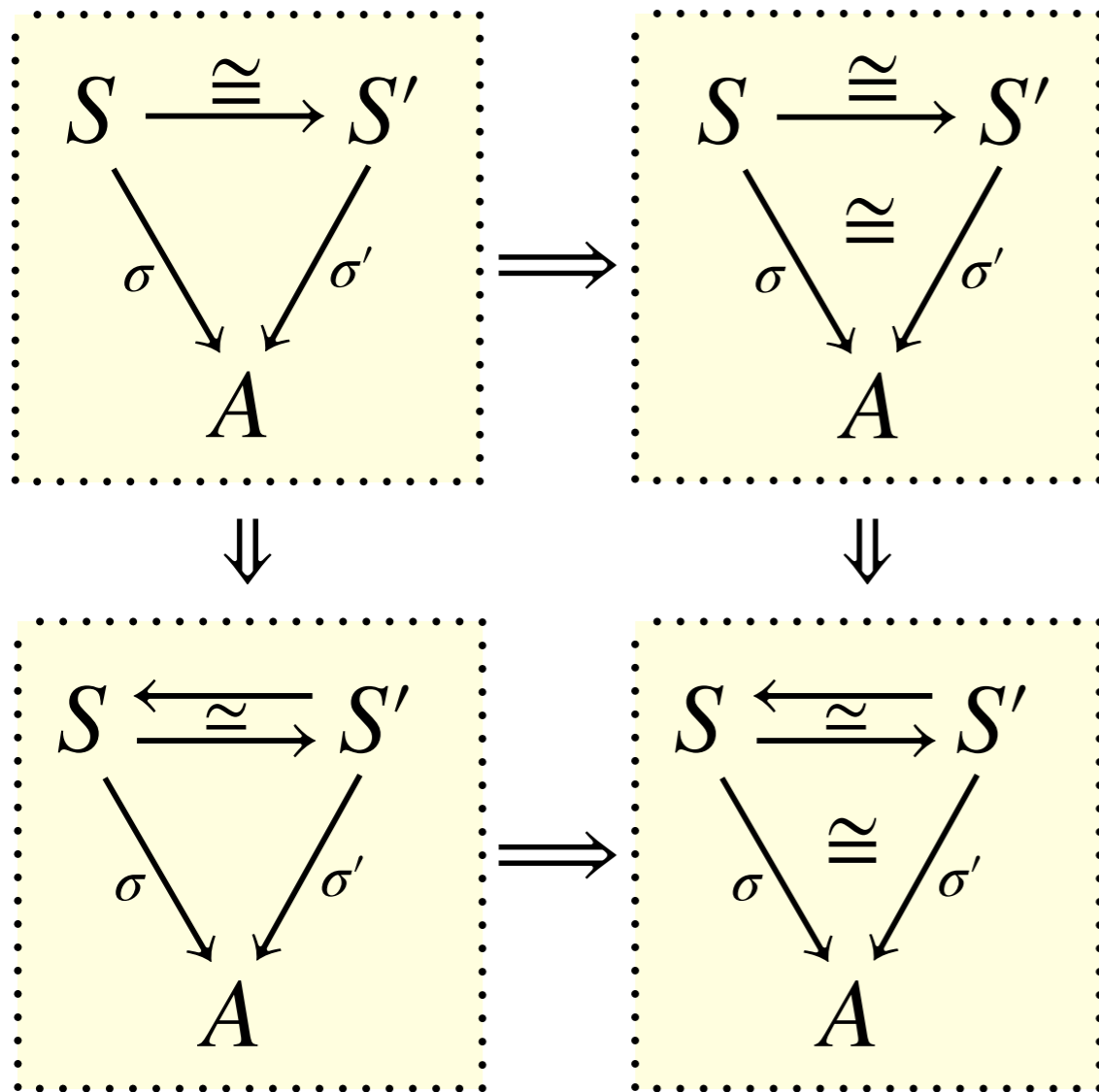
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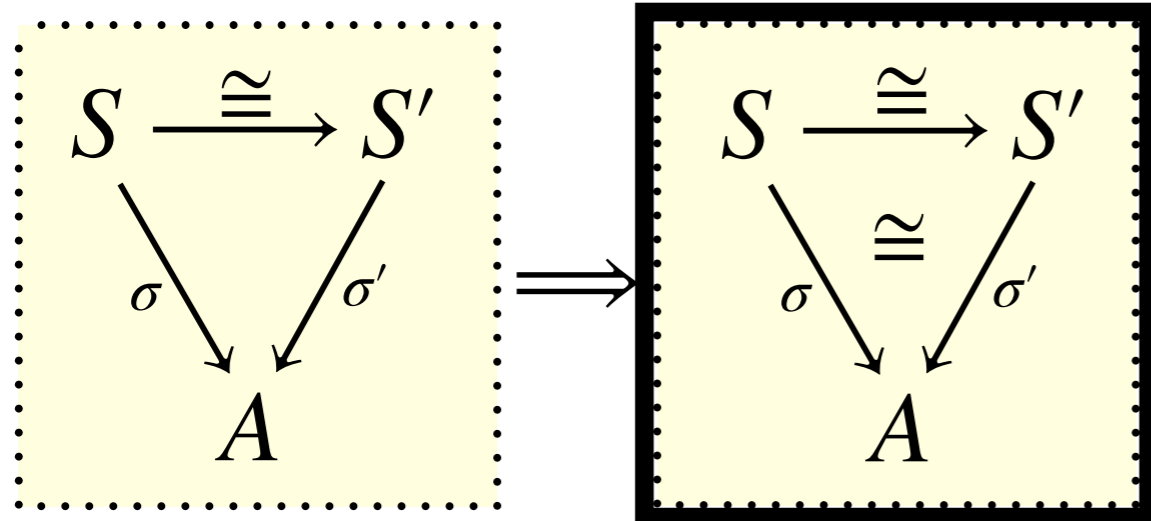
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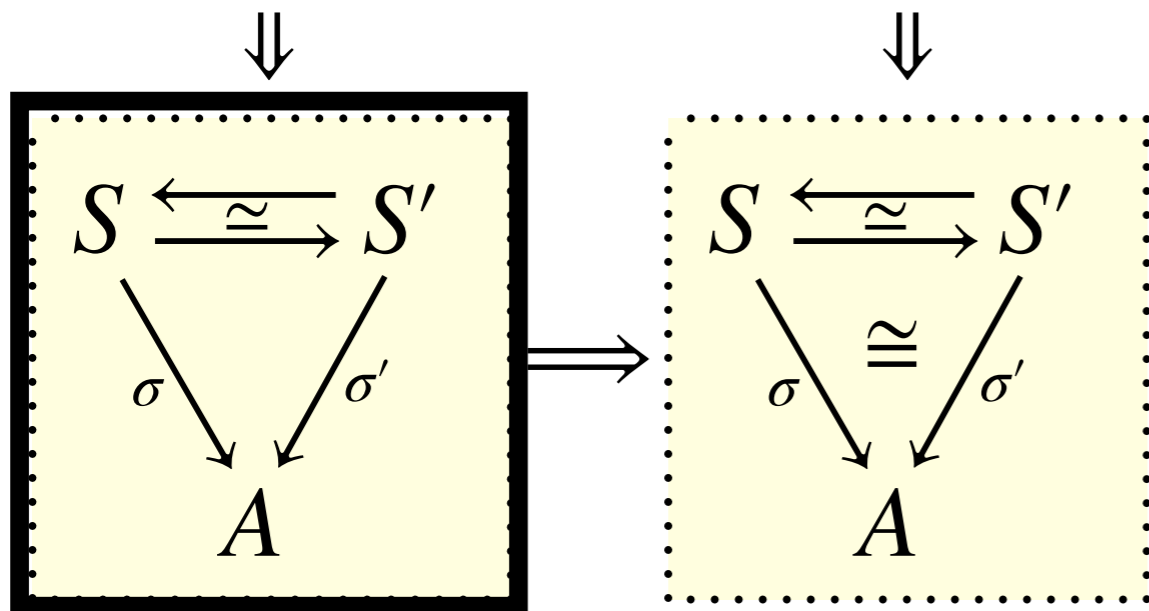
Choice of  $\sim$  (and  $\text{id}$ ,  $\circ$ )  
determines a model:

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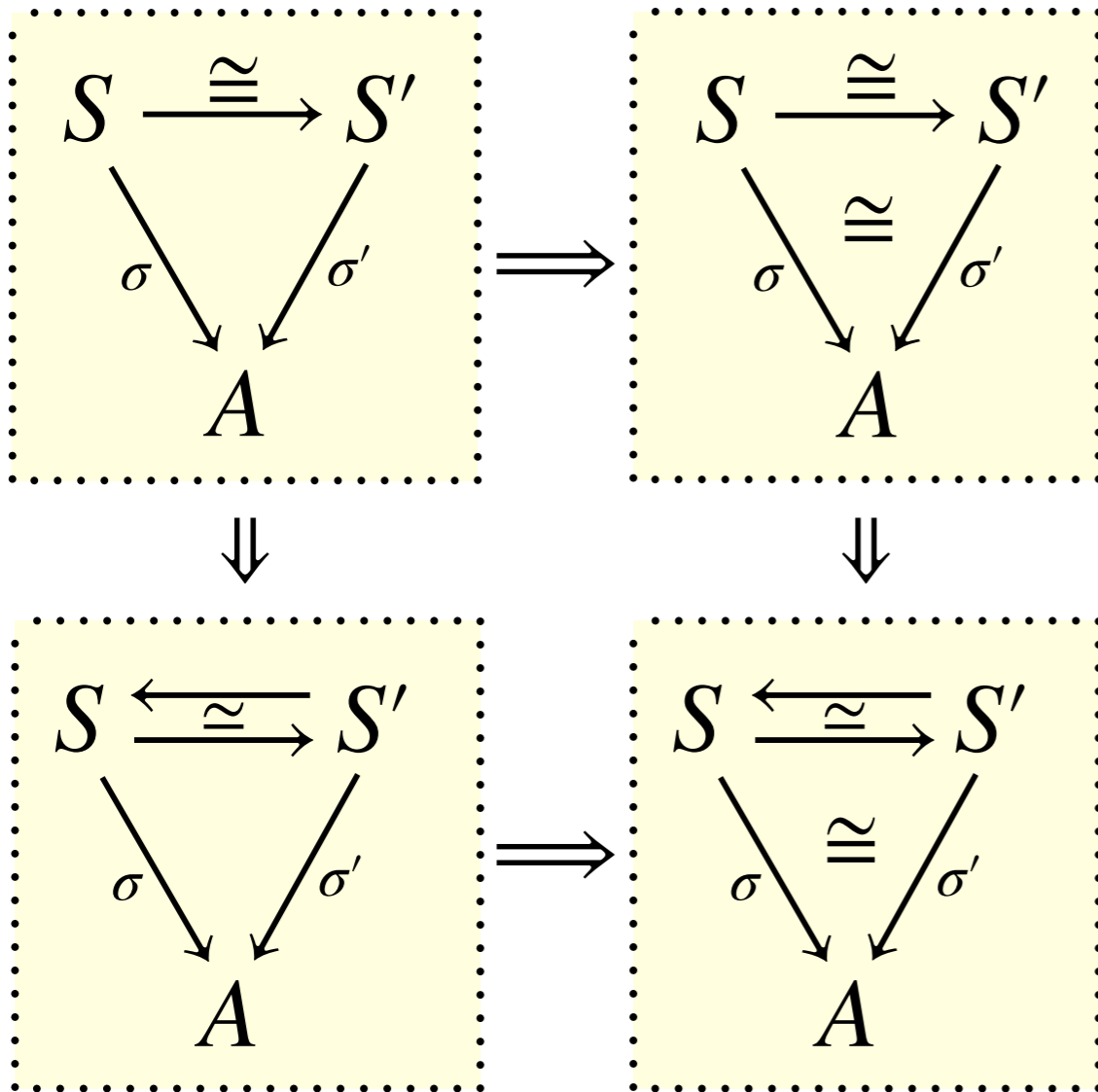


*Thin concurrent games*  
 [Clairambault and others, 2015–now]

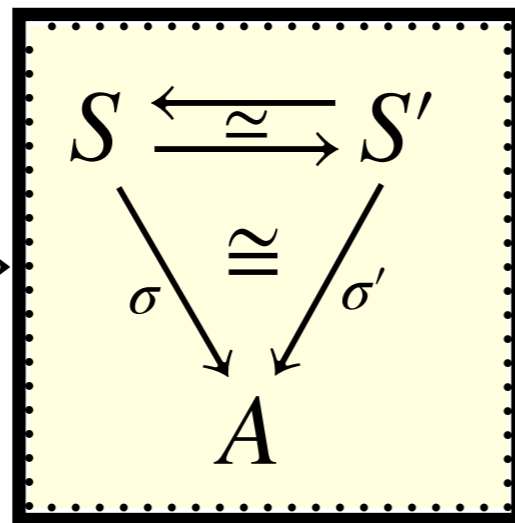
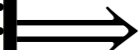
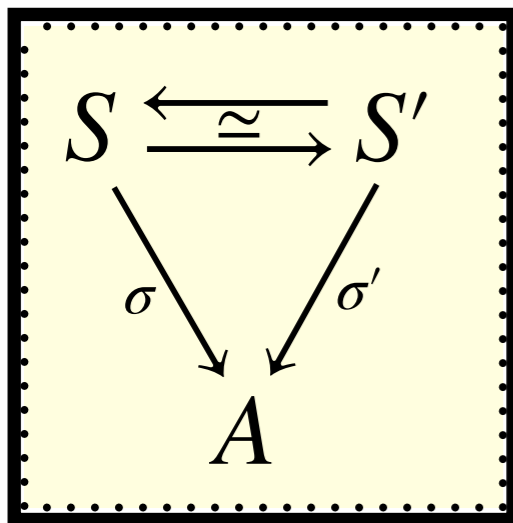
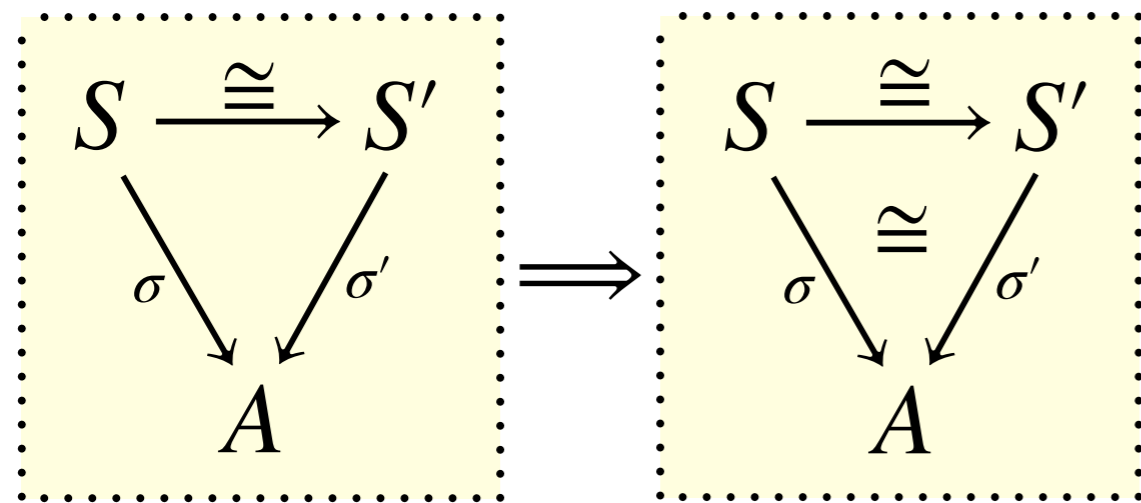


[CCW 2014]

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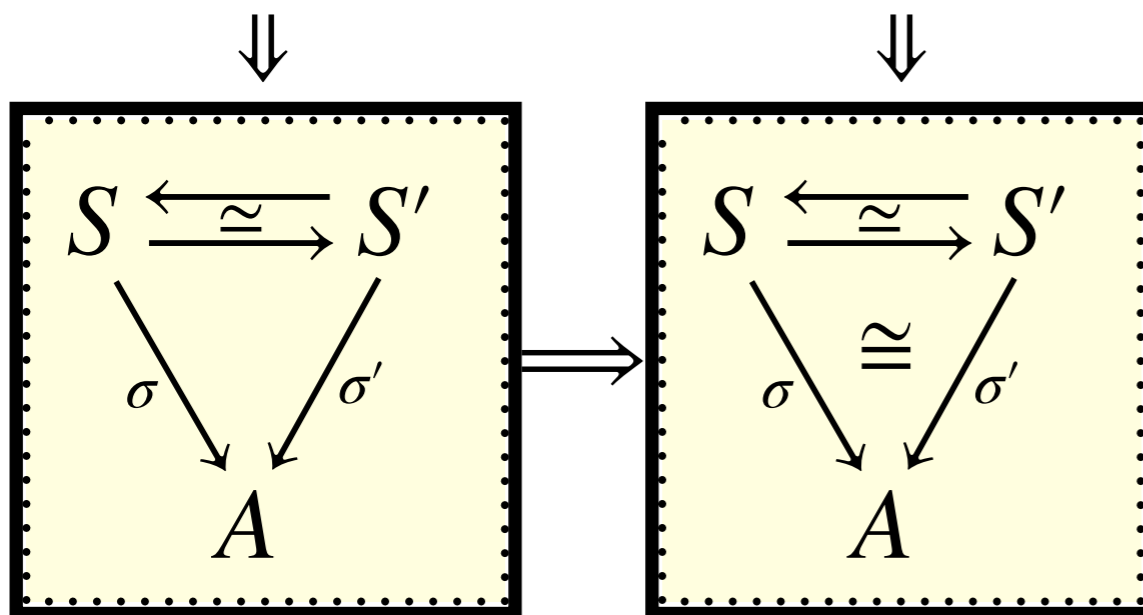
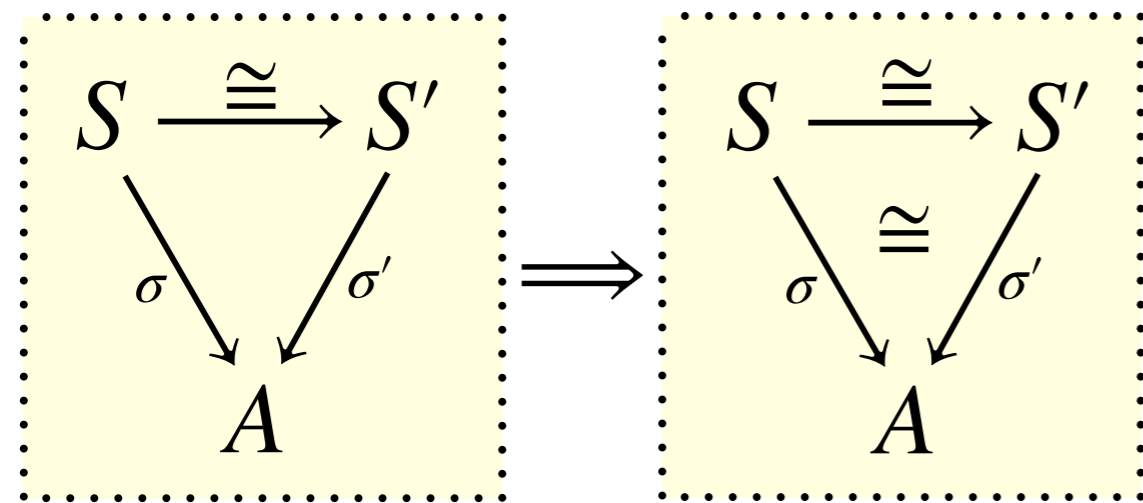
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*Strong  
strategies*

*Weak  
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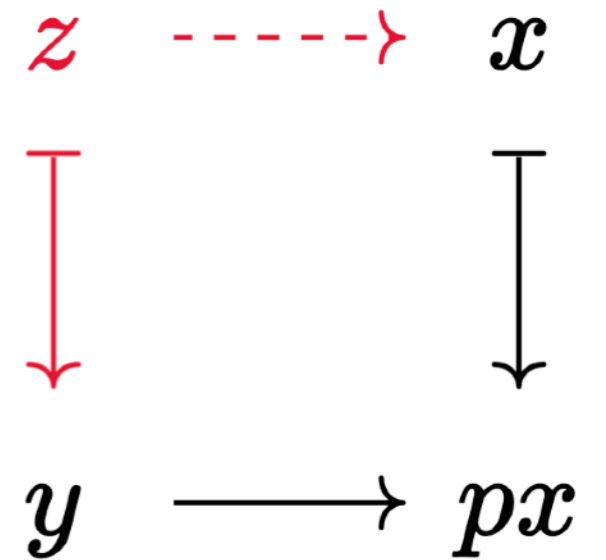
*Weak  
strategies*

We characterized these classes of strategies in terms of lifting properties.

# Lifting properties

A functor  $p : \mathbb{C} \rightarrow \mathbb{D}$  is a

- a **discrete fibration** if it has **unique** lifts
- a **Grothendieck fibration** if it has **cartesian** lifts
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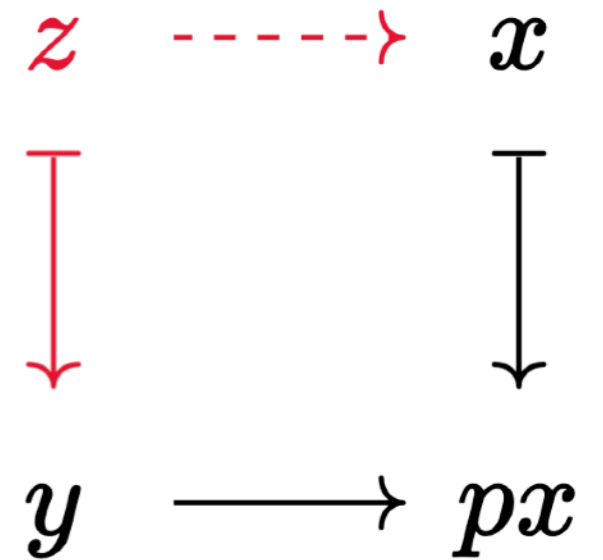
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$$\begin{array}{ccc} z & \overset{\text{---}}{\dashrightarrow} & x \\ \downarrow & & \downarrow \\ w & \overset{\cong}{\dashrightarrow} & y \longrightarrow px \end{array}$$

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*(because all fibres are setoids)*

# Main theorem.

*$\sigma$  is a weak strategy*

$$\text{id}_B \circ \sigma \circ \text{id}_A \simeq \sigma$$



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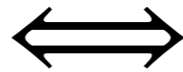
collapse construction

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# Two more results.

1. Representation of strategies as profunctors.

$$\sigma : \mathcal{C}(\sigma) \longrightarrow \mathcal{C}(A^\perp \parallel B) \xrightarrow{\cong} \mathcal{C}(A)^{\text{op}} \times \mathcal{C}(B)$$

This is a Street fibration so essential fibres determine a pseudo-functor:

$$\sigma : \mathcal{C}(A) \times \mathcal{C}(B)^{\text{op}} \longrightarrow \mathbf{Setoid}$$

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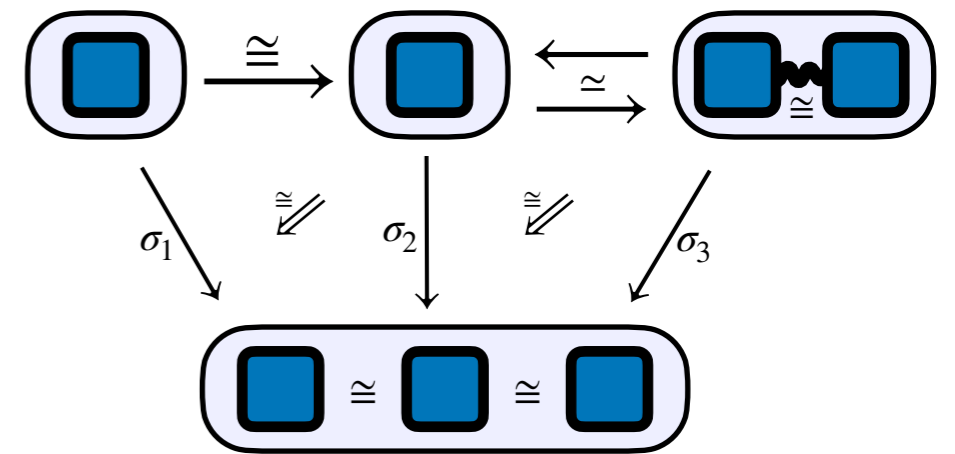
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2. Embedding of thin concurrent games  $\mathbf{Thin} \hookrightarrow \mathbf{Strat}$   
*(relevant to many applications in game semantics.)*

# Summary



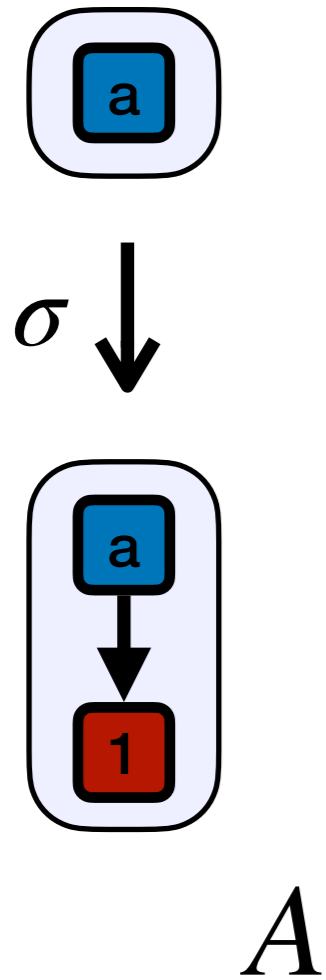
Categorical foundations for games and strategies on event structures with symmetry.

**2013:** Concurrent strategies as discrete fibrations

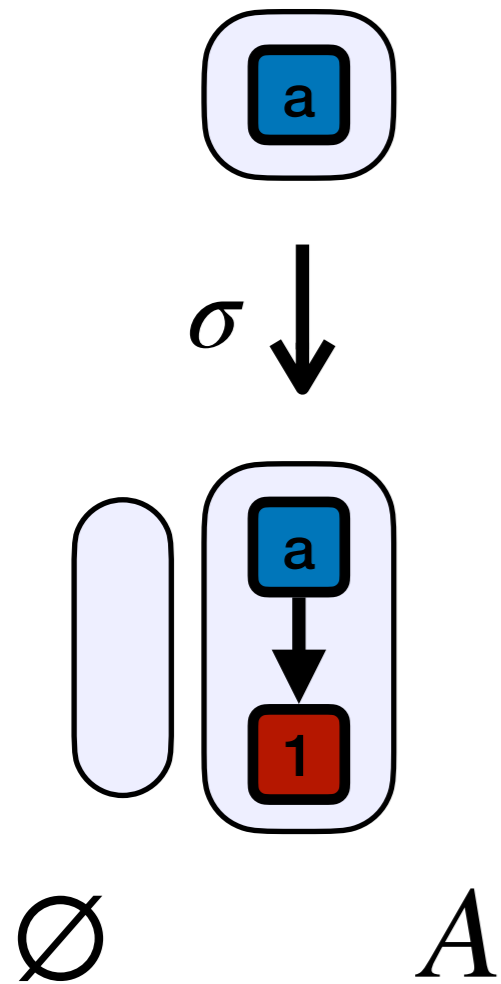
**Now:** Concurrent strategies with symmetry as Street fibrations.

All reasoning up to symmetry/homotopy. Maybe better for a type-theoretic formalization?

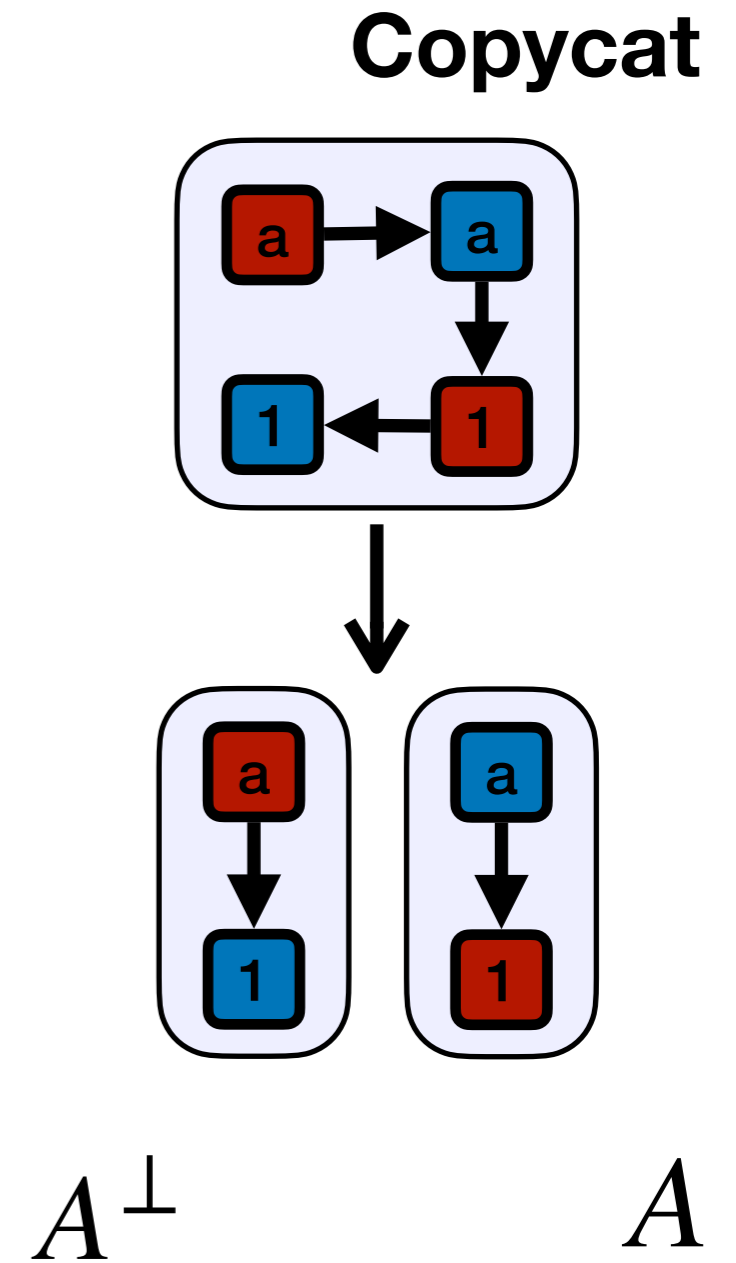
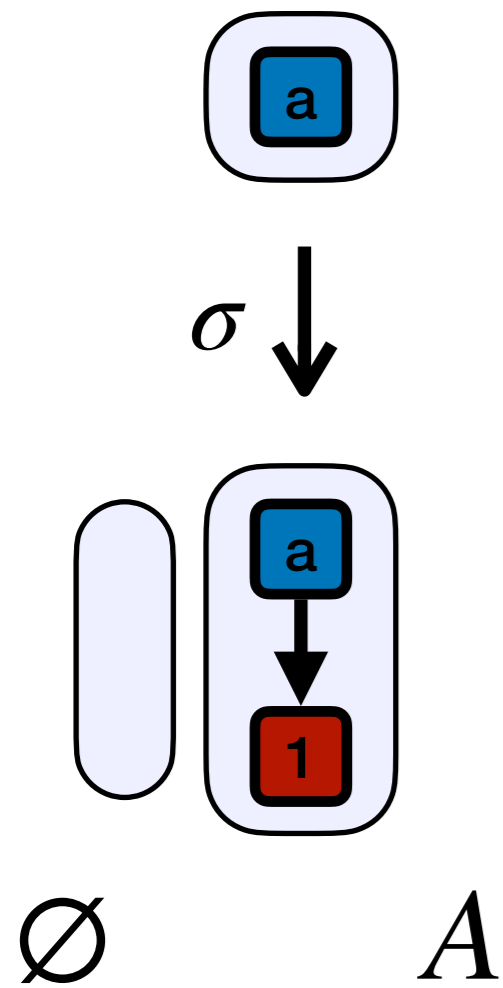
*Example of a pre-strategy which is not a strategy:*



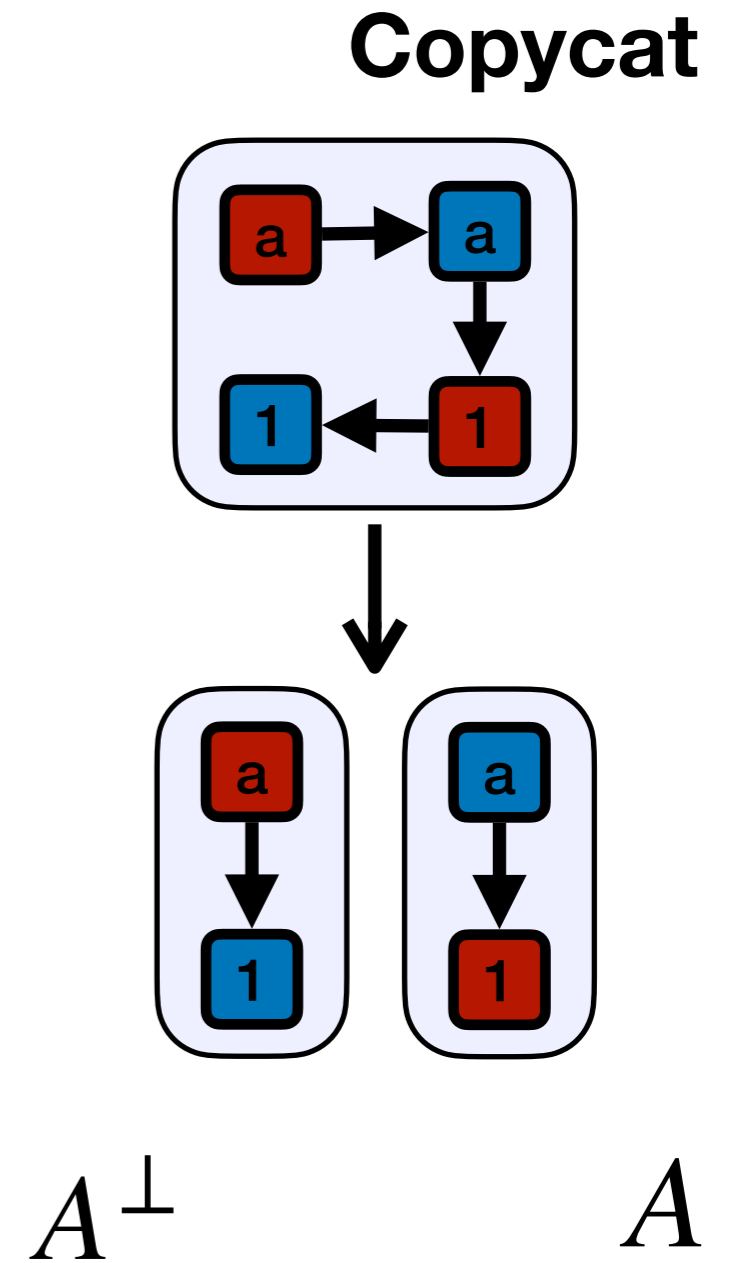
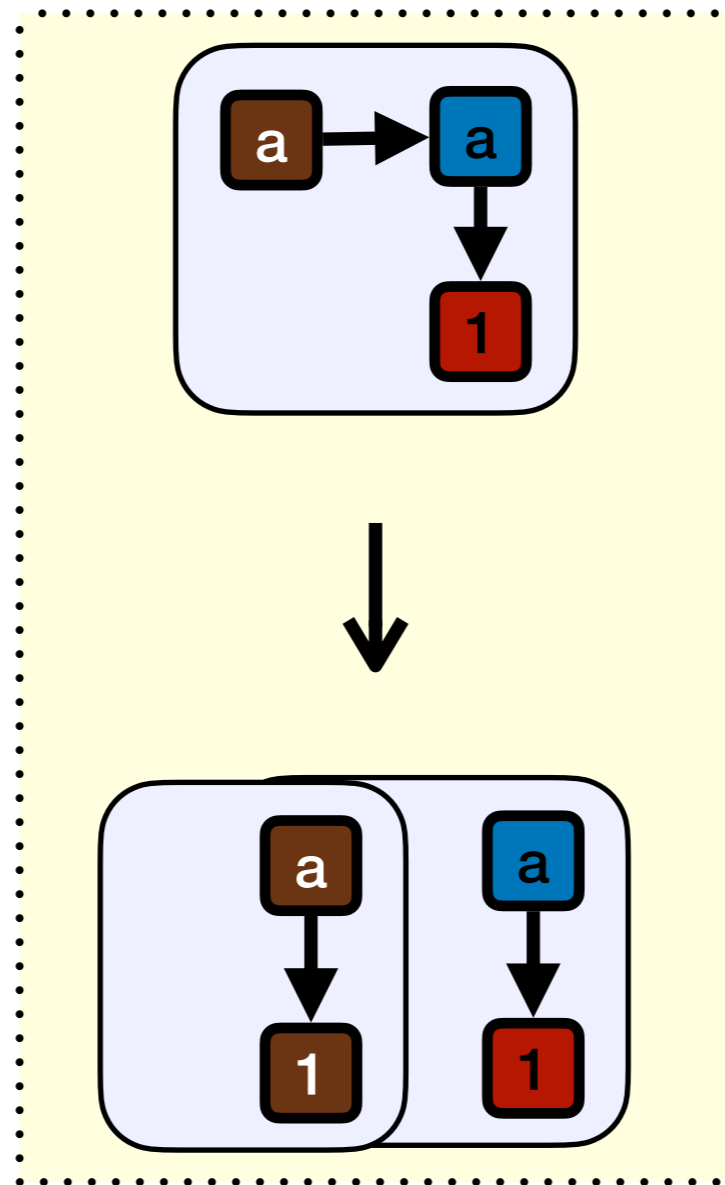
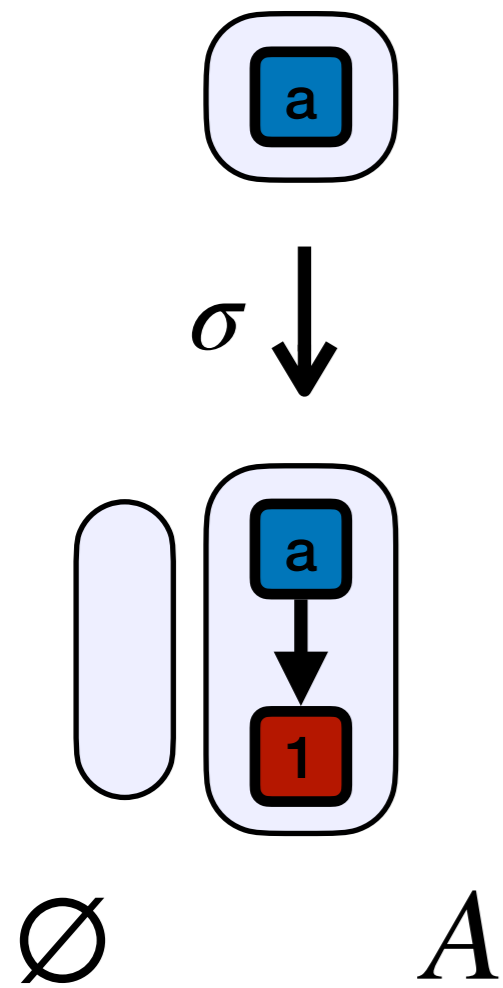
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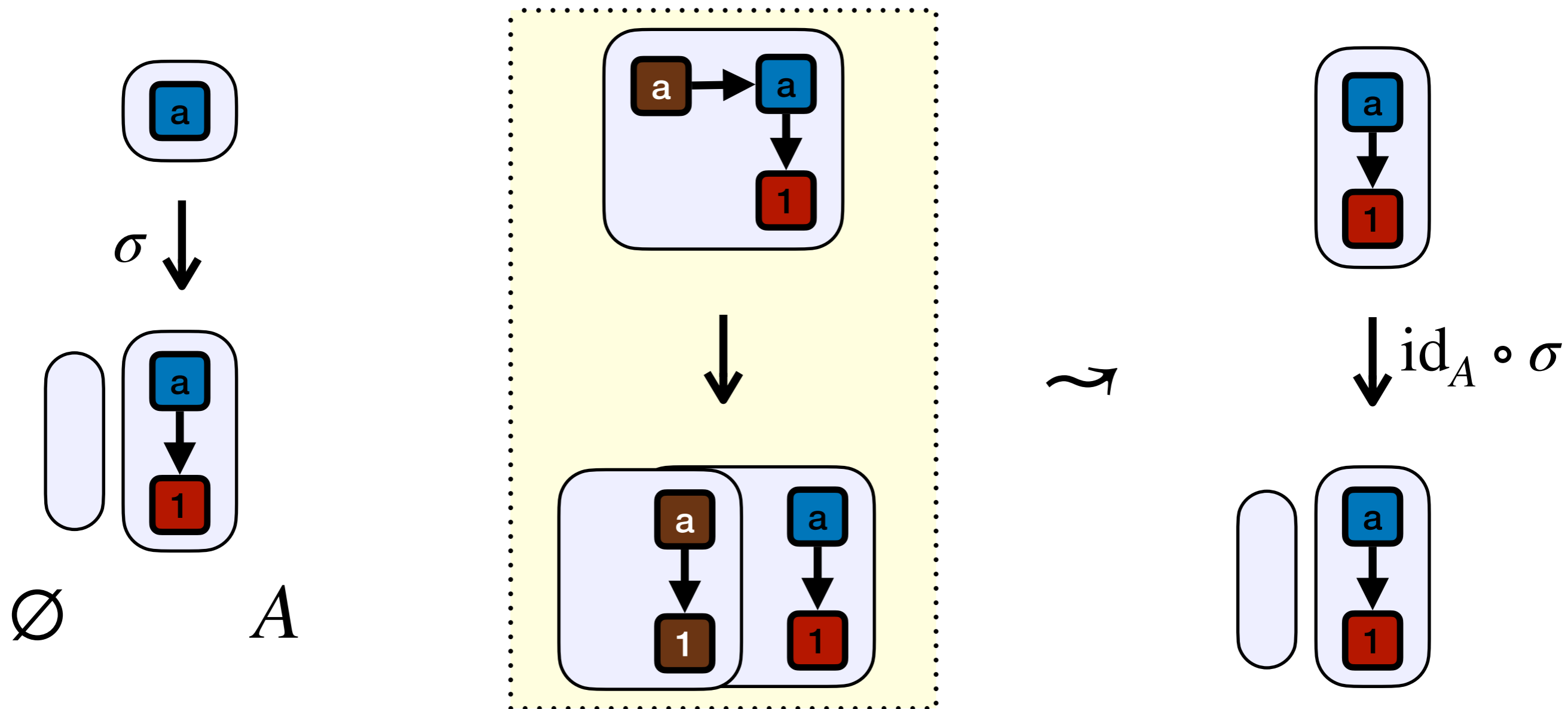
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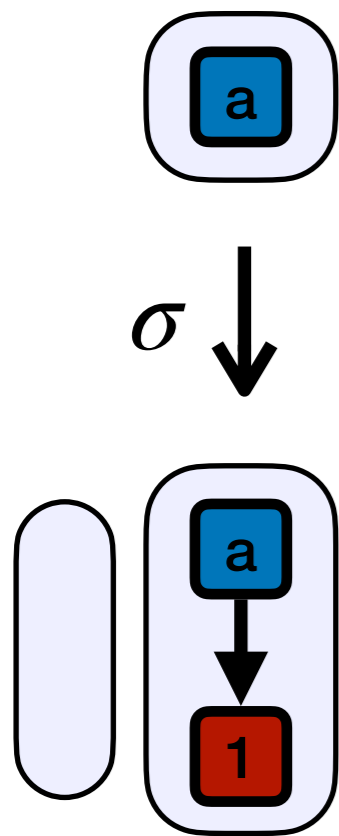
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$\neq$

