Algebraization of Language Metatheory

The Case of Coeffects

Francesco Gavazzo

University of Padua

Logical Aspects of Abstract Syntax

$\Gamma \vdash \mathtt{t} \rhd \mathtt{T}$

$\mathtt{t} \longrightarrow \mathtt{s}$

$\langle \sigma, \mathtt{t} \rangle \Downarrow \mathtt{v}$

$\phi \vdash \psi$

$\mathtt{t} \gtrapprox \mathtt{s}$

$\mathtt{p} \Vdash \phi$

Logical Predicates
on
Abstract Syntax

$\forall \mathtt{t}, \mathtt{s}, \mathtt{T}.\ (\mathtt{t} \rhd \mathtt{T}\ \&\ \mathtt{t} \longrightarrow \mathtt{s}) \supset \mathtt{s} \rhd \mathtt{T}$

$\forall \mathtt{t}, \mathtt{s}_1, \mathtt{s}_2.\ (\mathtt{t} \Downarrow \mathtt{s}_1\ \&\ \mathtt{t} \Downarrow \mathtt{s}_2) \supset \mathtt{s}_1 = \mathtt{s}_2$

$\forall \mathtt{t}, \mathtt{s}.\ \mathtt{t} \gtrapprox \mathtt{s} \supset \mathtt{C}[\mathtt{t}] \gtrapprox \mathtt{C}[\mathtt{s}]$

confluence, normalization

type safety, congruence

cut elimination , parametricity

. . .

Logical Properties
(☞ **Language Metatheory**)

A **general** theory of these logical aspects using **algebra** (**Algebraic Logic**)

- NATURAL ALGEBRA OF STRUCTURAL PREDICATES
- ALGEBRAIC ACCOUNT OF LOGICAL AND SYNTACTIC STRUCTURE
- SYNTAX-INDEPENDENT LANGUAGE METATHEORY

This Talk:

- Overview of **algebras of predicates over term structures**
- Extensions to **quantitative** and **graded predicates** (☞**coeffects**)

Algebras of Abstract Predicates

BOOLE, DE MORGAN, PEIRCE, SCHRÖDER, ...

LOGIC = ALGEBRAS OF PREDICATES OVER A DOMAIN OF OBJECTS

$$\underbrace{W}_{\text{UNIVERSE OF DISCOURSE}} \rightsquigarrow \underbrace{\mathrm{Pred}(W)}_{\text{ALGEBRAIC STRUCTURE}}$$

- $a \leq b$ LOGICAL ENTAILMENT
- $a \wedge b$, $a \vee b$, .. LOGICAL FORM

$$\boxed{\mathrm{Pred}(W) \triangleq \mathrm{Rel}(W, W)}$$

ALGEBRA OF PREDICATES = ALGEBRA OF RELATIONS

DIAGONAL	Δ	$\{(x, x) \mid x \in W\}$
CONVERSE	a°	$\{(y, x) \mid (x, y) \in a\}$
COMPOSITION	$a ; b$	$\{(x, z) \mid \exists y.\ (x, y) \in a \ \&\ (y, x) \in b\}$
IMPLICATION	$a \rightarrow b$	$\{(w, v) \mid \forall u.\ (u, w) \in a \supset (u, v) \in b\}$
	$b \leftarrow a$	$\{(w, v) \mid \forall u.\ (v, u) \in a \supset (w, u) \in b\}$
ITERATION	a^{*}	$\bigcup_{n \in \mathbb{N}} a^{(n)}$

RELATION ALGEBRA(S) = ABSTRACT ALGEBRAS OF RELATIONS

$$\underbrace{\mathcal{A}}_{\text{ABSTRACT RELATIONS}} + \underbrace{\{\wedge, \vee, \dots, \Delta, ;, (\,)^\circ, \dots\}}_{\text{RELATION OPERATIONS}} + \underbrace{\begin{array}{c} (a \to a)^* = a \to a \\ a^* = \Delta \vee a ; a^* \\ \vdots \end{array}}_{\text{ALGEBRAIC LAWS}}$$

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EXPRESSIVENESS

CONFLUENCE	${}^*\!\!\leftarrow ; \to^* \;\leq\; \to^* ; {}^*\!\!\leftarrow$
EQUIVALENCE	$\Delta \leq \simeq, \quad \simeq^\circ \leq \simeq, \quad \simeq ; \simeq \leq \simeq$
PROGRESS	$\lhd ; \to \;\leq\; \lhd$
WF-INDUCTION	$x \leq x ; a \implies x \leq \bot$

$\rightsquigarrow$ CHURCH-ROSSER, NEWMAN'S LEMMA, ...

BUT NOT EXPRESSIVE ENOUGH

- OPERATIONS TO FORM COMPOUND LOGICAL PREDICATES, BUT NOT FOR THE STRUCTURE OF 'ATOMIC' PREDICATES

$$\left.\begin{array}{c} t \rightarrow s \\ \langle t, \sigma \rangle \Downarrow v \\ t \gtrapprox s \\ \varphi \vdash \psi \\ \vdots \end{array}\right\}$$

DEFINED USING TERM STRUCTURE

- STRUCTURAL RECURSION
- PATTERN MATCHING
- SUBSTITUTION

$\vdots$

BUT NOT EXPRESSIVE ENOUGH

- OPERATIONS TO FORM COMPLEX LOGICAL PREDICATES, BUT NOT FOR THE STRUCTURE OF 'ATOMIC' PREDICATES

$$\left.\begin{array}{c} t \longrightarrow s \\ \langle t, \sigma \rangle \Downarrow v \\ t \gtrsim s \\ \varphi \vdash \psi \\ \vdots \end{array}\right\}$$

DEFINED USING TERM STRUCTURE

- STRUCTURAL RECURSION
- PATTERN MATCHING
- SUBSTITUTION

$\vdots$

- CAN EXPRESS LOGICAL PROPERTIES, BUT NOT STRUCTURAL ONES (e.g. BISIMILARITY IS A CONGRUENCE)

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$\vdots$

- CAN EXPRESS LOGICAL PROPERTIES, BUT NOT STRUCTURAL ONES (e.g. BISIMILARITY IS A CONGRUENCE)

- CAN ALGEBRAIZE LOGICAL REASONING, BUT NOT STRUCTURAL ONE

RA = ALGEBRAS OF ABSTRACT PREDICATES

$Rel(W, W) \rightsquigarrow$ NO REFERENCE TO (THE STRUCTURE OF) W

ALGEBRAS OF STRUCTURAL PREDICATES

$Struct \rightsquigarrow Pred(Struct)$

→ LOGICAL FORM

→ STRUCTURAL FORM

Algebras of Term Predicates

GENERAL APPROACH

STRUCTURE $\mathcal{S}$ $\rightsquigarrow$ $\mathrm{Pred}(\underbrace{\cup \mathcal{S}}_{\text{CARRIER}})$ + OPERATIONS

INTERNALISE STRUCTURE AS ALGEBRAIC OPERATIONS ON PREDICATES

STRUCTURAL REASONING $\hookrightarrow$ LOGICAL REASONING

EXAMPLES

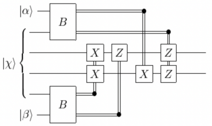
$$\underbrace{(W, R)}_{\text{RELATIONAL STRUCTURE}} \rightsquigarrow \underbrace{\mathrm{Prop}(W) + \{\Box, \Diamond\}}_{\text{MODAL LOGIC}}$$

$$\underbrace{(W, \cdot, 1)}_{\text{MONOID}} \rightsquigarrow \underbrace{\mathrm{Prop}(W) + \{\bullet, 1, \backslash, /\}}_{\text{LAMBEK CALCULUS}}$$

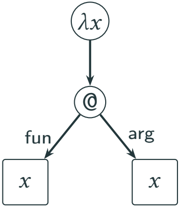
$$\underbrace{(W, *)}_{\text{PAIR STRUCTURE}} \rightsquigarrow \underbrace{\mathrm{Rel}(W) + \{\ldots, \Delta, ;, \ldots, \nabla\}}_{\text{FORK ALGEBRA}}$$

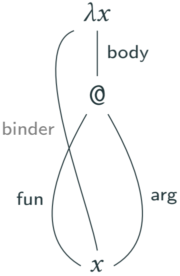
What About Term Structure?

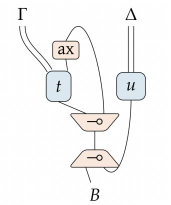
$\lambda x.x(x)$ $\qquad \prod_{x:A}.\,\mathtt{refl}(x) : x =_A x$ $\qquad (\sum_{n\geq 0} 2^n)+\int_a^b e^{ax}\mathrm{d}x$

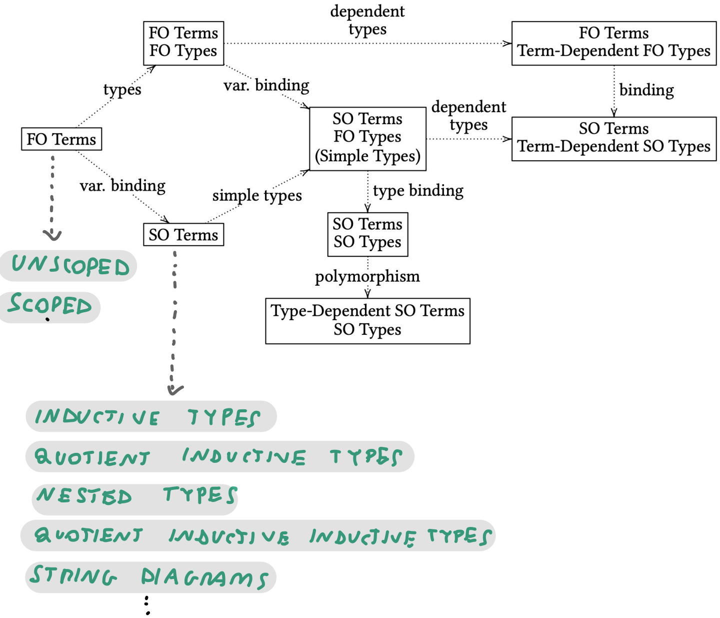


Abstract Syntax

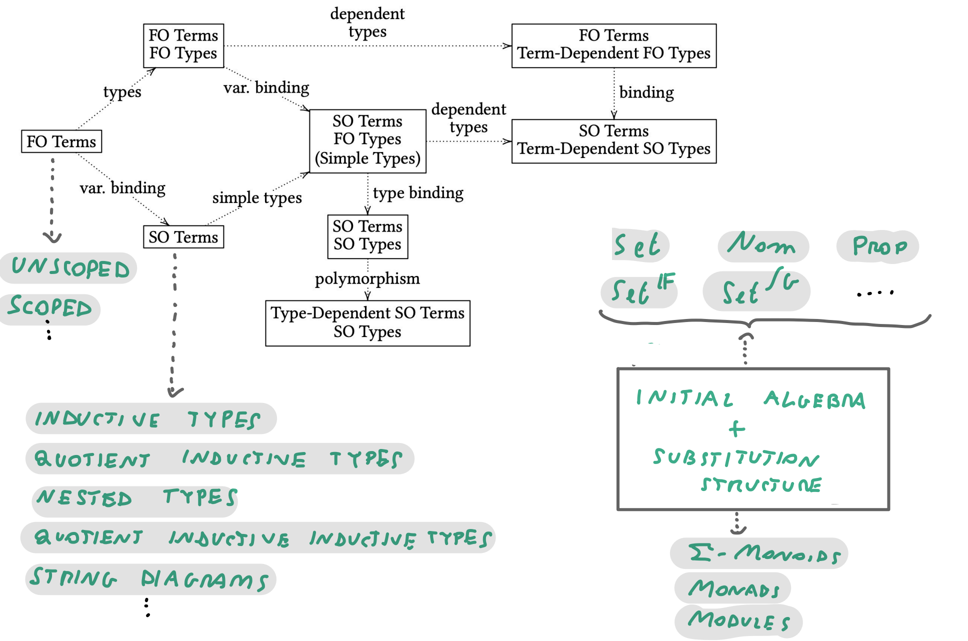







dependent types
FO Terms
FO Types
FO Terms
Term-Dependent FO Types
types
var. binding
binding
FO Terms
SO Terms
FO Types
(Simple Types)
dependent types
SO Terms
Term-Dependent SO Types
var. binding
simple types
type binding
SO Terms
SO Terms
SO Types
polymorphism
Type-Dependent SO Terms
SO Types
UNSCOPED
SCOPED
INDUCTIVE TYPES
QUOTIENT INDUCTIVE TYPES
NESTED TYPES
QUOTIENT INDUCTIVE INDUCTIVE TYPES
STRING DIAGRAMS
⋮

WHAT ABOUT TERM STRUCTURE?



MAIN RESULT, I [LICS 2023]

(i) 'ALL' TERM STRUCTURES INDUCE ESSENTIALLY THE SAME ALGEBRA OF RELATIONS

(ii) THIS ALGEBRA FAITHFULLY INTERNALISE THE SYNTAX STRUCTURE OF TERMS AS ALGEBRAIC OPERATIONS

(iii) THIS ALGEBRA CAN BE AXIOMATISED AS AN ABSTRACT ALGEBRA OF (TERM) RELATIONS

Term Relation Algebras (TRAs)

Variables $\Delta_\eta \leq \Delta$

Operations $\widetilde{a;b} = \tilde{a};\tilde{b}$ $\quad \tilde{a}^\circ = \widetilde{a^\circ}$ ⋮ $\Big\}$ Relator

Recursion $a^\dagger$ unique solution $x = \hat{x};a$ $\quad \Delta^\dagger = \Delta$ $\Big\}$ Initial algebra

Substitution

$a[\Delta_\eta] = a = \Delta_\eta[a]$ $\quad a[b[c]] = (a[b])[c]$ $\Big\}$ Monoidal

$a[b] \leq c \iff a \leq b \gg c$ $\}$ Monoidal closed

$\tilde{a}[b] \leq \widetilde{a[b]}$ $\}$ Strength

COMMENTS

- TRAS ARISE CANONICALLY FROM SYNTAX STRUCTURES
- THEY EXPLAIN MANY OPERATIONAL CONSTRUCTIONS [LASSEN]

a^+ $\left\{ \begin{array}{l} \text{HOWE'S EXTENSION} \\ \text{PARALLE REDUCTION} \end{array} \right\}$ INITIAL ALGEBRA

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- TRAS ARE SYNTAX - INDEPENDENT

$\leadsto$ DEEP INVARIANT OF SYNTAX (GIRARD)

$\leadsto$ ALL TRA THEOREMS HOLD ON ANY TERM STRUCTURE

COMMENTS

- TRAs ARISE CANONICALLY FROM SYNTAX STRUCTURES
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- TRAs ARE SYNTAX - INDEPENDENT

⇝ DEEP INVARIANT OF SYNTAX (GIRARD)

⇝ ALL TRA THEOREMS HOLD ON ANY TERM STRUCTURE

WHAT KIND OF THEOREMS CAN WE PROVE?

MAIN RESULT, II [LICS 2026]

(i) WITHIN TRAs, WE CAN DEFINE

- PARALLEL & SEQUENTIAL REDUCTION
- BIG-STEP SMALL-STEP OP. SEMANTICS
- APPLICATIVE (BI)SIMILARITY
- SEMANTIC EQUIVALENCES (KLEENE, CONTEXTUAL, ...)

EX. PARALLEL REDUCTION $a^{\Rightarrow} \triangleq (a[\Delta] \vee \Delta)^{\dagger}$

MAIN RESULT, II [LICS 2026]

(ii) WE CAN MODEL FUNDAMENTAL SEMANTIC PRINCIPLES

- GENTZEN INVERSION PRINCIPLE
- GENTZEN CONSERVATION PRINCIPLE

$\boxed{\text{Ex. (GIP)} \qquad a = \langle \bar{\Delta} \rangle ; a}$

MAIN RESULT, II [LICS 2026]

(iii) WE CAN PROVE FUNDAMENTAL METATHEOREMS

$$\underbrace{a \models \text{DET, GIP, GCP}}_{\text{LOCAL CONDITIONS}} \Longrightarrow \underbrace{\left(\begin{array}{l} a^{\Downarrow} \models \text{DET} \\ a^{\Rightarrow} \models \text{CONFLUENT} \\ a^{\simeq} \models \text{CONGRUENCE} \end{array} \right)}_{\text{GLOBAL PROPERTIES}}$$

MAIN RESULT, II [LICS 2026]

(iii) APPLICATIONS TO

- λ- CALCULUS
- PCF
- $\Lambda 2$
- $\lambda\Pi$
- $\mathfrak{Z}$-CALCULUS
- $\Lambda 2 \mu$
- Δ - CALCULUS
- CL

Towards Coeffects

QUANTITATIVE & GRADED PREDICATES

$\Gamma \vdash t \triangleright T \rightsquigarrow r \cdot \Gamma \vdash t \triangleright [s]\, T$

$t \to s \rightsquigarrow t \to_{\varepsilon} s$

$\langle \sigma, t \rangle \Downarrow v \rightsquigarrow \langle \sigma, t \rangle \Downarrow_{[s_1, \ldots, s_n]} v$

$\phi \vdash \psi \rightsquigarrow \varphi \vdash^{w} \psi$

$t \gtrapprox s \rightsquigarrow \delta^{\gtrapprox}(t, s)$

$p \Vdash \phi \rightsquigarrow p \Vdash_{\varepsilon} \varphi$

Logical Predicates on **Abstract Syntax** $\rightsquigarrow$ **LOGICAL PREDICATES** ON **RESOURCE-SENSITIVE ABSTRACT SYNTAX**

Core Idea

(1) Algebras of Quantitative Relations

- $\mathcal{R}el(W) = \underbrace{\Omega^{W \times W}}_{\text{Quantale-valued metrics}}$
- $\mathcal{R}el(W) = \underbrace{\mathcal{P}(W \times \mathcal{R} \times W)}_{\text{Weighted relations}}$

Relation Algebras

Core Idea

(2) Algebras of Quantitative Relations over Resource Syntax Structure

Internalise this as

Graded Relational Modalities

$$[r * s](a ; b) \leq [r]a ; [s]b$$

$$[r][s]a \leq [r * s]a$$

$$[r]\tilde{a} \leq \widetilde{[r]a}$$

Conclusion

$\Gamma \vdash \mathtt{t} \rhd \mathtt{T}$

$\mathtt{t} \longrightarrow \mathtt{s}$

$\langle \sigma, \mathtt{t} \rangle \Downarrow \mathtt{v}$

$\phi \vdash \psi$

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$\forall \mathtt{t}, \mathtt{s}.\ \mathtt{t} \gtrapprox \mathtt{s} \supset \mathtt{C}[\mathtt{t}] \gtrapprox \mathtt{C}[\mathtt{s}]$

confluence, normalization

type safety, congruence

cut elimination , parametricity

...

Logical Properties
(☞ **Language Metatheory**)

ALGEBRA

$\Gamma \vdash t \triangleright T$

$t \longrightarrow s$

$\langle \sigma, t \rangle \Downarrow v$

$\phi \vdash \psi$

$t \gtrapprox s$

$p \Vdash \phi$

$\forall\, t, s, T.\ (t \triangleright T \ \&\ t \longrightarrow s) \supset s \triangleright T$

$\forall\, t, s_1, s_2.\ (t \Downarrow s_1 \ \&\ t \Downarrow s_2) \supset s_1 = s_2$

$\forall\, t, s.\ t \gtrapprox s \supset C[t] \gtrapprox C[s]$

confluence, normalization

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. . .

Logical Predicates
on
Abstract Syntax

Logical Properties
(☞ **Language Metatheory**)

ALGEBRA

ALGEBRAIC SEMANTICS

LOGICAL SYSTEMS

METATHEORY

FORMS OF JUDGMENTS

INFERENCE RULES