An introduction to Graded Logics, Graded Types, and their Semantics

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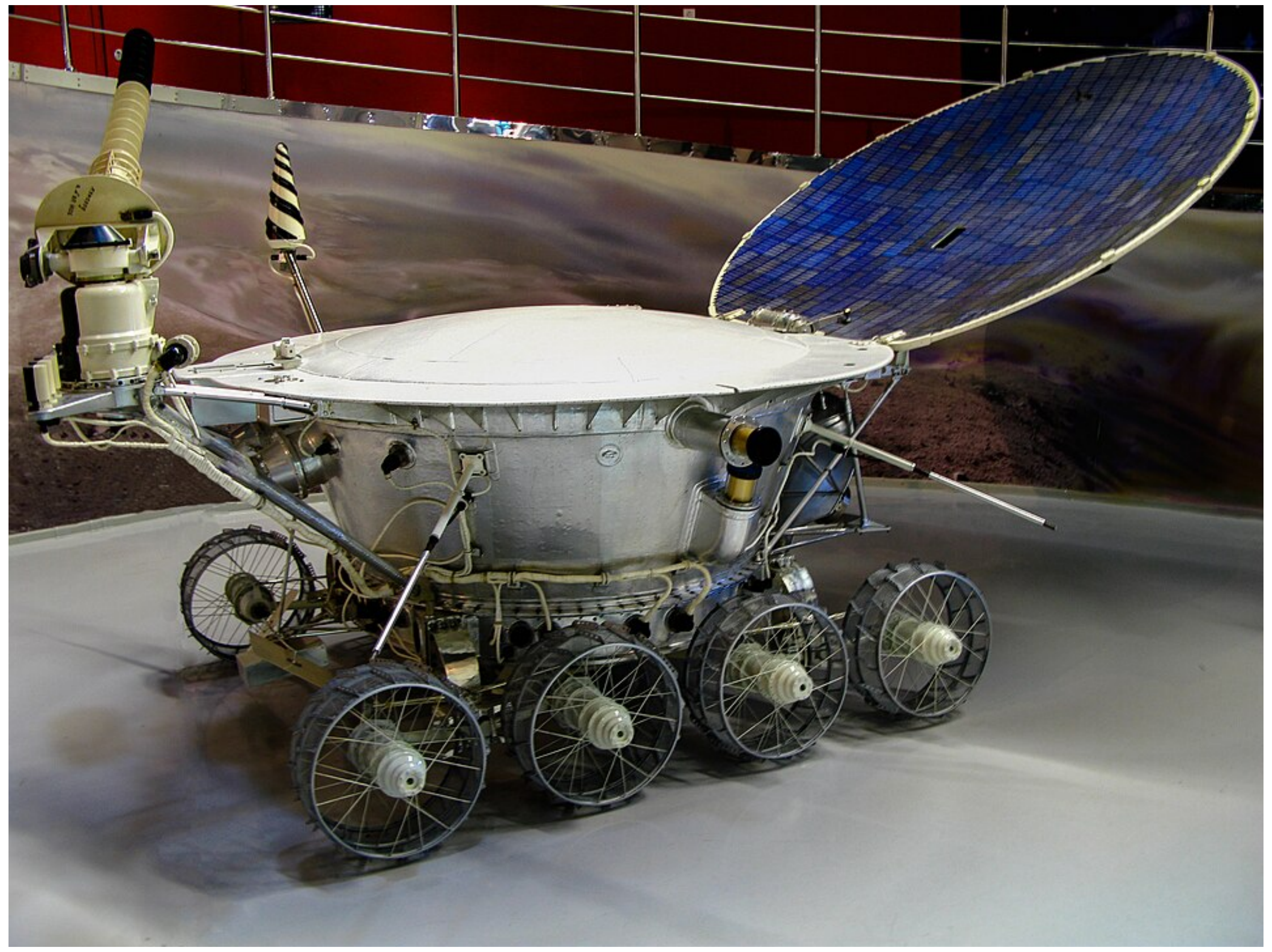
UNIVERSITY OF CAMBRIDGE

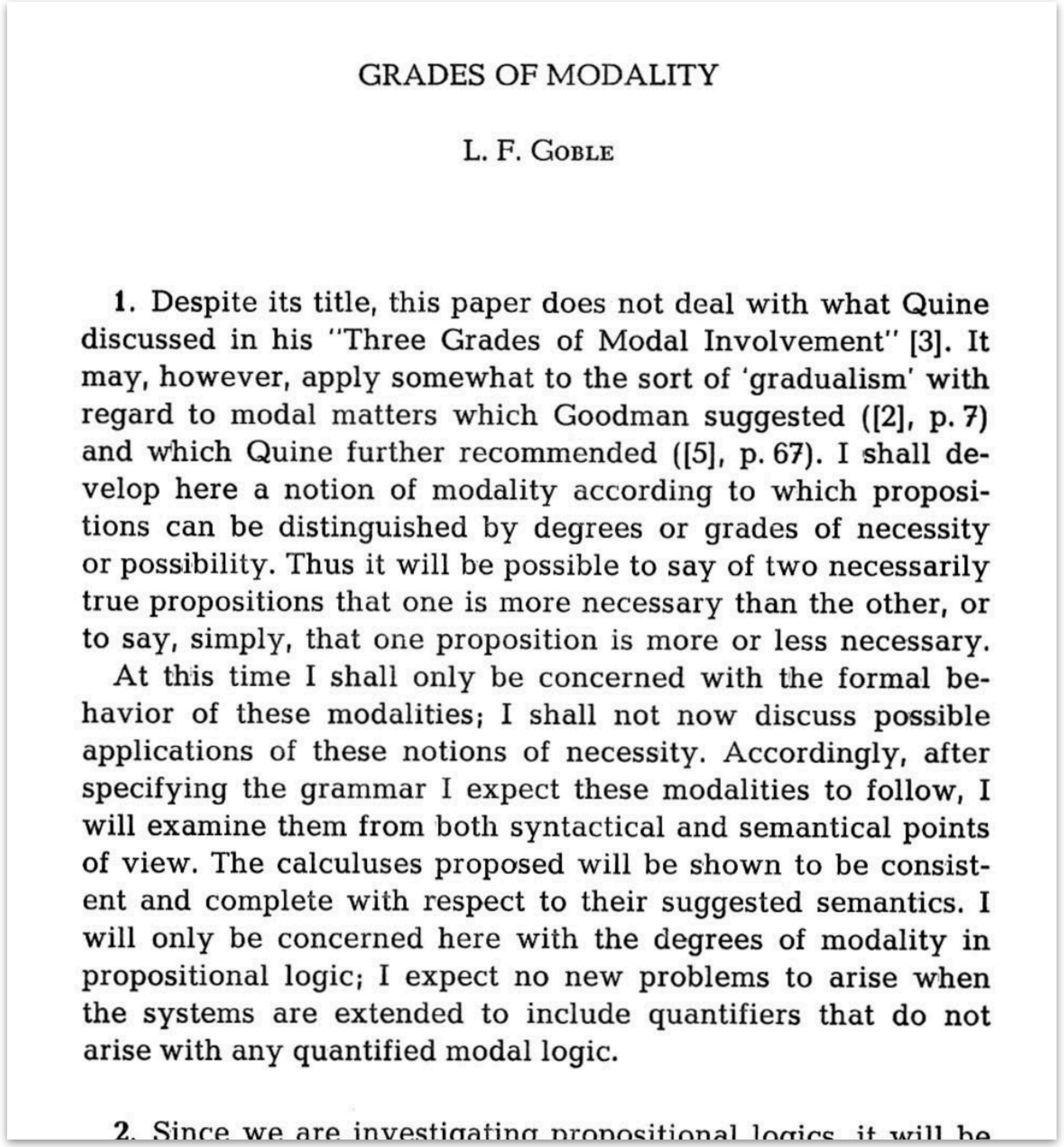
University of Kent

1970

Apollo 13 mission



Soviet Lunokhod rover drives on the moon



GRADES OF MODALITY

L. F. Goble

1. Despite its title, this paper does not deal with what Quine discussed in his "Three Grades of Modal Involvement" [3]. It may, however, apply somewhat to the sort of 'gradualism' with regard to modal matters which Goodman suggested ([2], p. 7) and which Quine further recommended ([5], p. 67). I shall develop here a notion of modality according to which propositions can be distinguished by degrees or grades of necessity or possibility. Thus it will be possible to say of two necessarily true propositions that one is more necessary than the other, or to say, simply, that one proposition is more or less necessary.

At this time I shall only be concerned with the formal behavior of these modalities; I shall not now discuss possible applications of these notions of necessity. Accordingly, after specifying the grammar I expect these modalities to follow, I will examine them from both syntactical and semantical points of view. The calculuses proposed will be shown to be consistent and complete with respect to their suggested semantics. I will only be concerned here with the degrees of modality in propositional logic; I expect no new problems to arise when the systems are extended to include quantifiers that do not arise with any quantified modal logic.

2. Since we are investigating propositional logics, it will be

What if one formula is *more necessary* than another?

Grades of Modality (Goble, 1970)

“Degrees of necessity” - Logic $\mathscr{L}_k$

- Indexed family of modalities $\{N_i A\}_{i \in [1..k]}$

  e.g. $N_2 A$ is “more necessary” than $N_1 A$

- **Gradation:** $(j \leq i) \;\Rightarrow\; N_i A \to N_j A$
- **Nec:** $\vdash A$ *implies* $\vdash N_i A$
- **(K)** $N_i(A \to B) \to N_i A \to N_i B$
- **(T)** $N_i A \to A$
- **(4)** $N_i A \to N_i N_i A$ (**a modern observation**: ordering induces $N_{i \sqcup j} A \to N_i N_j A$)

- *Model*: indexed sets of ‘necessary’ propositions connected by $[1..k]$-family of accessibilities relations

In So Many Possible Worlds (Fine, 1972)

Quantifying worlds (between $\exists$ of $\Diamond$ and $\forall$ of $\Box$)

- Indexed family of modalities $\{\Diamond^{\geq i} A\}_{i \in [0, |\mathscr{W}|]}$

  meaning A is true in at least i accessible worlds
- Model $(\mathscr{W}, R \subseteq \mathscr{W} \times \mathscr{W})$ with $w \vDash \Diamond^{\geq i} A$ **iff** $|\{v \in \mathscr{W} \mid w\,R\,v \wedge v \vDash A\}| \geq i$
- **Gradation:** $(j \leq i) \Rightarrow \Diamond^{\geq i} A \rightarrow \Diamond^{\geq j} A$
- **Base:** $\Diamond^{\geq 0} A \leftrightarrow \top$
- ***Additional axioms depending on properties of R*** *(not considered by Fine), e.g.,*
  - **(T)** $A \rightarrow \Diamond^{\geq 1} A$ (for reflexive R)

A general view (so far): pre-ordered graded modalities

Syntax

- Indexing set I with preorder $(\leq) \subseteq I \times I$
- Indexed family of modalities $\{F_i : \mathbb{P} \to \mathbb{P}\}_{i \in I}$
- Gradation: $(j \leq i) \;\Rightarrow\; F_i A \to F_j A$

Semantics

- A contravariant functor $F : (I, \leq)^{\mathrm{op}} \to [\mathbb{P}, \mathbb{P}]$

A common philosophy: *make (modal) reasoning more fine-grained via **grading***

(Goble) po-graded necessity, (Fine) po-graded possibility

Type-and-effect systems (Gifford, Lucassen, 1986)

Make typing more fine-grained to capture *intensional aspects* (side effects)

Typing judgments $\Gamma \vdash e : \tau, F$ include description of side effects

$$\frac{\Gamma, x : \sigma \vdash e : \tau, F}{\Gamma \vdash \lambda x \,.\, e : \sigma \xrightarrow{F} \tau, \emptyset} \qquad \frac{\Gamma \vdash e_1 : \sigma \xrightarrow{H} \tau, F \quad \Gamma \vdash e_2 : \sigma, G}{\Gamma \vdash e_1 \; e_2 : \tau, F \cup G \cup H} \qquad \frac{(x : \sigma) \in \Gamma}{\Gamma \vdash x : \sigma, \emptyset}$$

Originally sets of effect operations;
Nielson and Nielson (’94,’99) generalisd to pre-ordered monoids $(\mathscr{M}, \bullet, 1, \sqsubseteq)$

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e.g., cost via: $(\mathbb{N}, +, 0, \leq)$

A graded type system *sans modalities*: annotation attached to judgments; structure of annotations reflects sequential semantics of effects

Graded monads model effect systems (Katsumata, 2014)

- Model of type-and-effect systems $[\![\Gamma \vdash t : A, F]\!] : [\![\Gamma]\!] \rightarrow T_F\ [\![A]\!]$ where T is **a graded monad:**
  - $T : \mathscr{M} \rightarrow [\mathscr{C}, \mathscr{C}]$ where $(\mathscr{M}, \bullet, 1)$ is a monoidal category
  - **unit** $\eta_A : A \rightarrow T_1 A$ and **multiplication** $\mu_{F,G,A} : T_G T_F A \rightarrow T_{F \bullet G} A$
  - associativity and unitality coherence conditions

- Or build a **graded monadic meta language with explicit** $T_F A$ **type constructor**

(see also O, Petricek, Mycroft, 2014)

Modal Type Analysis

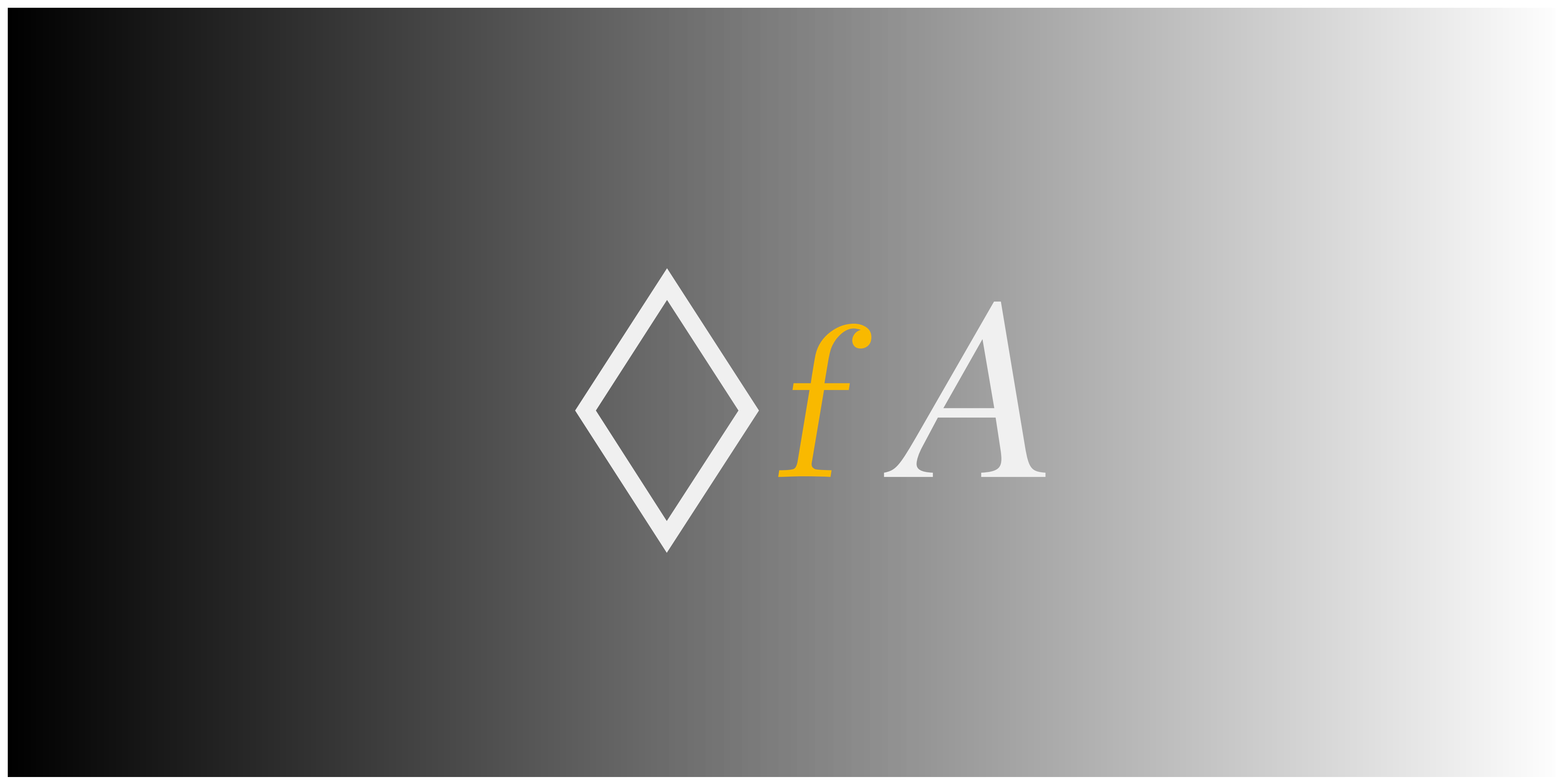
A

MA

Pure

Effectful

Graded Modal Type Analysis



$\Diamond f A$

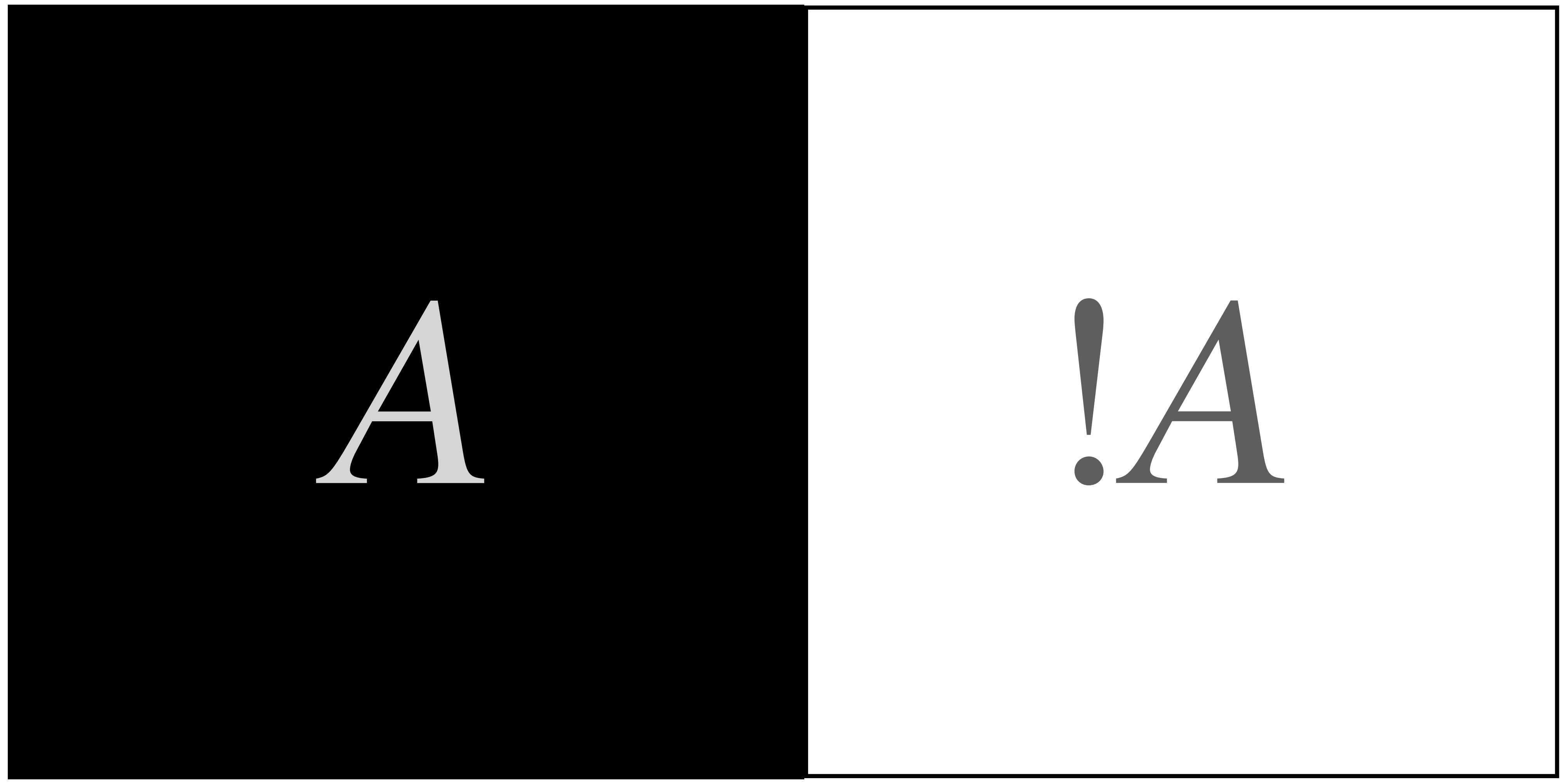
$f \in \mathscr{M}$

Monoid

Pure

Effectful

Modal Type Analysis



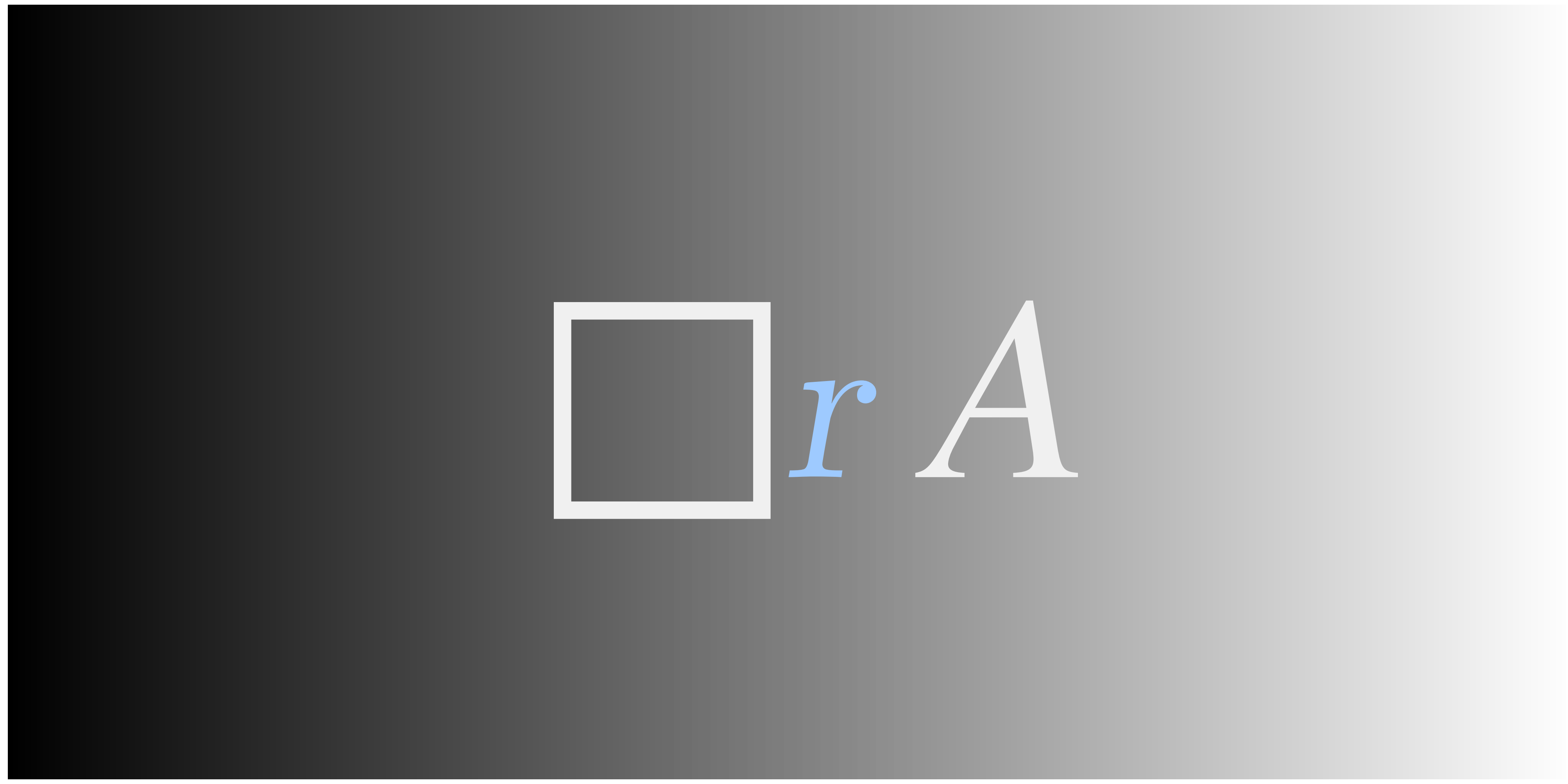
A

$!A$

linear

non-linear

Graded Modal Type Analysis



$\Box_r A$

linear

non-linear

Bounded Linear Logic (Girard, Scedrov, Scott, 1992)

('Rigid' subset here)

- Linear logic + $\mathbb{N}-$indexed family of modalities $!_r A$ meaning used at most r-times

$$\frac{!_{r_1}A_1,\ \dots,\ !_{r_n}A_n \vdash B}{!_{s*r_1}A_1,\ \dots,\ !_{s*r_n}A_n \vdash\ !_s B}\,pr \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\,der$$

$$\frac{\Gamma,\ !_x A,\ !_y A \vdash B}{\Gamma,\ !_{x+y} A \vdash B}\,contr \qquad \frac{\Gamma \vdash B}{\Gamma,\ !_0 A \vdash B}\,weak \qquad \frac{\Gamma, !_r A \vdash B \quad r \leq s}{\Gamma, !_s A \vdash B}\,approx$$

- **(2013-14)** Generalise to **(pre-ordered) semiring** [+ some technicalities for sound substitution]

Bounded linear types in a resource semiring (Ghica, Smith)

A Core Quantitative Coeffect Calculus (Brunel, Gaboardi, Mazza, Zdancewic)

Coeffects: a calculus of context-dependent computation (Petricek, O, Mycroft)

BLL / GLL semantics via *graded exponential comonad*

$! : \mathscr{R}^{\mathsf{op}} \to [\mathscr{C}, \mathscr{C}]$ where $(\mathscr{R}, *, 1, +, 0, \leq)$ is a pre-ordered semiring

$$!_{r*s}A \to\ !_r!_sA$$
$$!_1A \to A$$

promotion and dereliction

$$!_sA \to\ !_rA$$
where $r \leq s$

approximation

$$!_0A \to 1$$
$$!_{r+s}A \to\ !_rA \otimes !_sA$$

weakening and contraction

$$!_rA \otimes !_rB \to\ !_r(A \otimes B)$$
$$1 \to\ !_r1$$

monoidality - promotion over multiple variables

Goble’s logic has graded comonadic semantics

Revisiting the 1970s logics through the modern gadgets

- $N_i A$ forms a graded comonad over $([1..k], \sqcup, 1, \geq)$

$$\varepsilon_A : N_1 A \to A \qquad \delta_{i,j,A} : N_{i \sqcup j} A \to N_i N_j A$$

Another semiring-graded necessity example: confidentiality and non-interference

(Gaboardi, Katsumata, O, Breuvart, Uustalu, 2016)
(Abel & Bernardy, 2020)

- Lattice $r \in \mathscr{L}$ with top $\top$ and bottom $\bot$

- $\Box_r A$ means an A values that can **only** be "read" by "levels" $r \leq \ell$

(Liepelt, Marshall, O, Rajani, Vollmer, 2025)

- *Model*: graded exponential comonad $[\![-]\!] : \mathscr{L} \to [[\mathscr{L}, \Delta\mathbf{Tm}], [\mathscr{L}, \Delta\mathbf{Tm}]]$

$$[\![\Box_r A]\!]^{adv}_{\mathcal{V}} = \begin{cases} \{([t_1],[t_2]) \mid (t_1, t_2) \in [\![A]\!]^{adv}_{\mathcal{E}}\} & \mathit{when} \quad r \leq adv \\ \{([t_1],[t_2]) \mid t_1 \in [A]_{\mathcal{E}} \wedge t_2 \in [A]_{\mathcal{E}}\} & \mathit{when} \quad \neg(r \leq adv) \end{cases}$$

Theorem 3 (Non-Interference). *For all judgments* $x :_s A \vdash t : \Box_{adv}\mathbb{B}$ *where* $adv \leq s$ *and* $adv \neq s$ *then given* $\emptyset \vdash v_1 : A$ *and* $\emptyset \vdash v_2 : A$ *then* $[v_1/x]t \equiv [v_2/x]t$.

Graded linear logic + indexed types

In practice, with the Granule language implementation

https://granule-project.github.io

```
rep : forall {a : Type, n : Nat}
    . N n -> (a -> a) [n] -> (a -> a)
rep Z     [f] = \x -> x;
rep (S n) [f] = \x -> f (rep n [f] x)
```

```
copy : forall {a : Type, s : Semiring} . a [(1 + 1):s] -> (a, a)
copy [y] = (y, y)
```

Quantitative program reasoning with graded modal types
(O, Liepelt, Eades, ICFP 2019)

Where do they come from?
Graded (co)monads derived two ways

- (Bénabou, 1967) Monads are lax functors

$$T : [1,\mathbf{Cat}]_{\mathsf{lax}} \qquad T* = \mathscr{C} \qquad \eta : \mathsf{Id}_{\mathscr{C}} \Rightarrow Tid_{*} \qquad \mu : Tid_{*} \circ Tid_{*} \Rightarrow T(id_{*} \circ id_{*})$$

- Graded monads are (op)**lax functors from the delooping of a monoid**

$$T : [BM,\mathbf{Cat}]_{\mathsf{lax}} \qquad T* = \mathscr{C} \qquad \eta : \mathsf{Id}_{\mathscr{C}} \Rightarrow Tid \qquad \mu_{f,g} : Tg \circ Tf \Rightarrow T(g \circ f)$$

- Graded monads are (co)**lax monoidal functors**

$$T : (M, \bullet, I) \to ([\mathscr{C},\mathscr{C}], \circ, \mathsf{Id})$$

$$\begin{tikzcd}
1 \arrow[d, "1"'] \arrow[dr, bend left, "\mathsf{Id}"{name=U}] & \\
\mathcal{M} \arrow[r, "T"'] & {[\mathcal{C},\mathcal{C}]} \arrow[Rightarrow, from=U, to=2-1, "\eta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M} \times \mathcal{M} \arrow[r, "T \times T"] \arrow[d, "\otimes"'] & {[\mathcal{C},\mathcal{C}] \times [\mathcal{C},\mathcal{C}]} \arrow[d, "\circ"] \arrow[dl, Rightarrow, "\mu"] \\
\mathcal{M} \arrow[r, "T"'] & \mathcal{M} \times \mathcal{M}
\end{tikzcd}$$

(O, Wadler, Eades, MSFP 2020)

Graded monads for Live Variable Analysis (LVA)

Classical dataflow can be captured as a grade

- *Transfer functions* monoid $(\mathscr{P}(\mathbb{V}) \to \mathscr{P}(\mathbb{V}), \circ, id)$ [variable demand, out → in]
- *Graded state monad* $\mathsf{Gm}^{\psi} A$ *- a stateful computation with LVA transfer* ψ

$$\begin{aligned} \mathsf{get}_X &: \mathsf{Gm}^{gen(X)}\mathbb{Z} \\ \mathsf{put}_X &: \mathbb{Z} \to \mathsf{Gm}^{kill(X)}\ \top \end{aligned}$$

$$\begin{aligned} gen(X) &= \lambda d\,.\,d \cup \{X\} \\ kill(X) &= \lambda d\,.\,d \backslash \{X\} \end{aligned}$$

$$[\![\, l : X := Y + Z]\!]_{\mathsf{G}} = (g_l = \mathsf{do}\{x \leftarrow \mathsf{get}_Y;\ y \leftarrow \mathsf{get}_Z;\ \mathsf{put}_X(x+y)\})$$

$$: \mathsf{Gm}^{gen(Y) \circ gen(Z) \circ put(X)}\mathbb{Z} \ = \mathsf{Gm}^{\lambda d.(d \backslash \{X\}) \cup \{Y,Z\}}\mathbb{Z}$$

(Ivašković, Mycroft, O, FSCD 2020)

Graded monads for Live Variable Analysis (LVA)

Grades specialise the semantics

$$\text{State}\ A = \text{Store} \to A \times \text{Store}$$

$$\text{Gm}^{\psi} A = \text{Store}(\textit{liveIn}(\psi)) \to A \times \text{Store}(\textit{footprint}(\psi))$$

Read subset of store $\textit{liveIn}(\psi) = \psi(\emptyset)$

Provided subset of store $\textit{footprint}(\psi) = \psi(\emptyset) \cup (\mathbb{V} \backslash \psi(\mathbb{V}))$

Theorem - *Soundness of deadcode elimination*

Let $m : \text{Gm}^{\psi_1}\ 1, f : 1 \to \text{Gm}^{\psi_2} B$

If $(\mathbb{V} \backslash \psi_1(\mathbb{V})) \cap \psi_2(\emptyset) = \emptyset$ then $\mu(\text{Gm}^{\psi_1} f)m \equiv f()$

i.e., grades shows m is dead-code, which is validated denotationally

(Ivašković, Mycroft, O, FSCD 2020)

Sans modalities semantically: graded categories*

- A monoidal category $(\mathscr{M}, \bullet, 1)$ of **grades**
- A class Obj$(\mathscr{C})$ of objects
- **An $[\mathscr{M}, \mathbf{Set}]$-enriched category over $\mathscr{C}$ (via Day tensor/exponents)**
  - Hom object $\mathscr{C}(I, J)(m)$ for all objects $I, J \in$ Obj$(\mathscr{C})$ and $m \in \mathscr{M}$
  - Identity morphisms $id_I \in \mathscr{C}(I, I)(1)$
  - Composition $\circ : \mathscr{C}(J, K)(m) \times \mathscr{C}(I, J)(n) \to \mathscr{C}(I, K)(m \bullet n)$
  - Upcast function $\uparrow^n_m : \mathscr{C}(I, J)(m) \to \mathscr{C}(I, J)(n)$ for all grades $m \leq n$

**V-indexed categories,* Wood (1978)

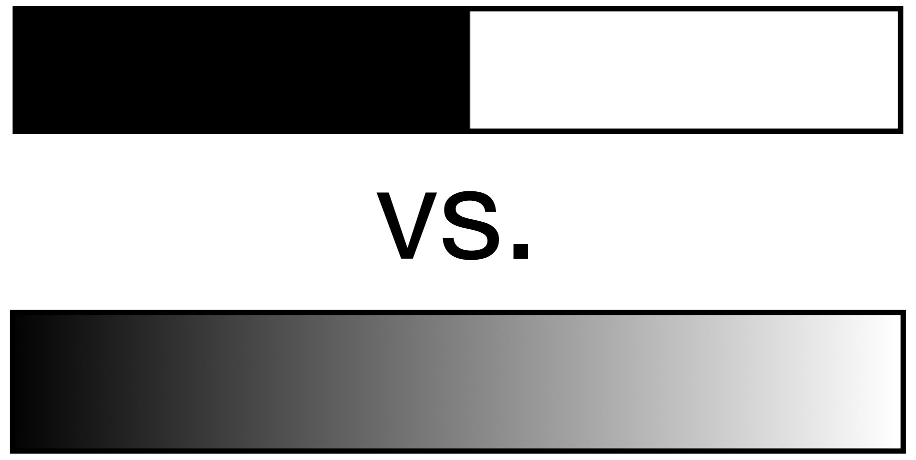
Locally graded categories, Levy (2019)

Graded Hoare Logic and its category semantics,
Gaboardi, Katsumata, O, Sato (2021)

Many more instances and applications…

- Subexponentials (Danos, Miller, Nigam)
- Contextual Model Type Theory (Nanevski, Ahmed, et al.)
- Hardware schedules (Ghica et al. ’14)
- Explicit provability logics (Artemov ’95, ’01)
- Quantitative operational models (Bianchini, Dagnino, Giannini, Zucca)
- Multi-stage programming (Pfenning & Davies, Zu)
- Quantitative Type Theory (McBride, Atkey)
- Costs (Cicek, Gaboardi, Rajani, Garg)
- Robustness / sensitivity (Gaboardi et al., Pierce et al.)
- Flexibly graded monads (McDermott, Katsumata, Uustalu, Wu)
- Degradings (McDermott)
- Contextads (Capucci, Myers)
- Adjoint logic (Hanukaev, Eades, Pfenning, Pruiksma)

Take home messages



- Graded structures enable fine-grained reasoning
  - Reason about additional properties in a logic/type ystem
  - Specialise a denotational model
  - Prune program synthesis (see: Hughes, O, (LOPSTR 2020, ESOP 2024))
- Formulated via some kind of lax functoriality

  functoriality = reflect structure of a proof theory / semantics in the grade
  (co)**laxness** = computational content

- **Models of polymorphic and dependent grades tricky**; lots to do here!

Thanks!