Approaching the Continuous from the Discrete

an Infinite Tensor Product Construction

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Contents

FinStoch and Motivation

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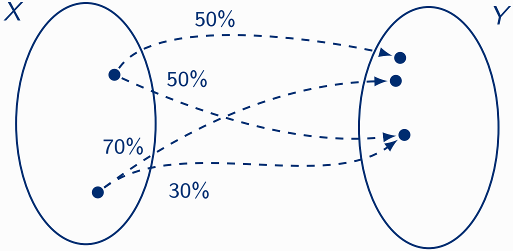

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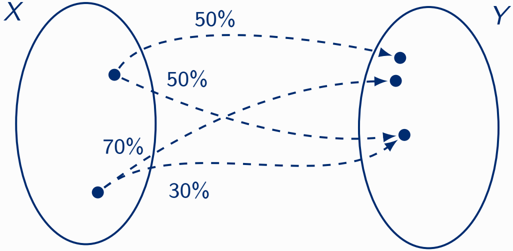

FinStoch comes equipped with a symmetric monoidal structure $(\otimes, I)$ (I is a singleton and the terminal object).

Motivation

Theorem

BorelStoch, the category of standard Borel spaces and Markov kernels, has *countably* infinite tensor products.[1]

[1] Fritz, T. and E. F. Rischel, Infinite products and zero-one laws in categorical probability, Compositionality, 2020.

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Spoiler: not really, but we do get probability measures.

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Infinite Tensor Products

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The infinite tensor product (ITP) $X = \bigotimes_{i \in J} X_i$ is the limit, preserved by $- \otimes Y$, of the diagram given by

$$\bigotimes_{i \in F} X_i \xrightarrow{\mathsf{id} \otimes \bigotimes_{i \in F \setminus F'} \mathsf{del}_{X_i}} \bigotimes_{i \in F'} X_i$$

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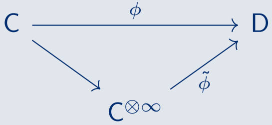
for all finite subsets $F' \subseteq F \subseteq J$. From now on, $\bigotimes_{i \in F} X_i = X_F$.

Universal property

Theorem

Let C be a semicartesian category.

There exist $C^{\otimes\infty}$ with all ITPs and $C \hookrightarrow C^{\otimes\infty}$ such that, for all $\phi\colon C \to D$ where D has all ITPs, there is $\tilde{\phi}\colon C^{\otimes\infty} \to D$ ITP-preserving satisfying



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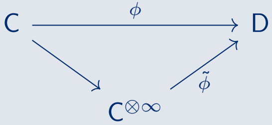

Idea of the proof

Take $[C, \mathsf{Set}]^{\mathrm{op}}$ restricted to the required limits and use Day convolution.

On description of morphisms

Morphisms in $\mathsf{C}^{\otimes\infty}$ can be described using limits of colimits:

$$\mathsf{C}^{\otimes\infty}(X, Y) \cong \lim_{G} \operatorname{colim}_{F} \mathsf{C}(X_F, Y_G)$$

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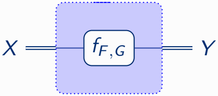
$\Rightarrow$ Finite approximation families: $f : X \to Y$ is such that for any finite G there is a finite F and a unique $X_F \to Y_G$ approximating f.

Plate Notation for Symmetric Monoidal Theories

Graphical notation

$f\colon X \to Y$ in $\mathsf{C}^{\otimes\infty}$ is described by the family $(f_{F,G}\colon X_F \to Y_G)_{F,G}$ in C.
We then write



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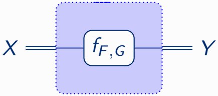
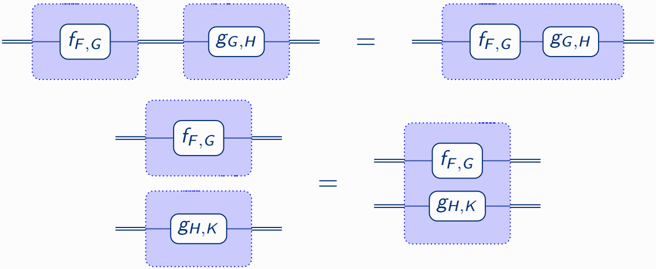


Remark

If C has cancellative deletions, $f_{F,G}$ is uniquely determined by f.
Otherwise, the same discussion applies but one has to be slightly more careful.

Equations for plate notation



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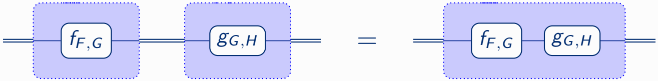
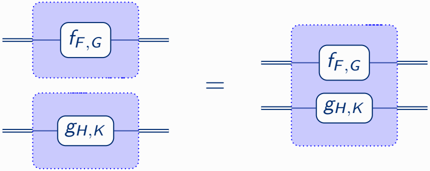
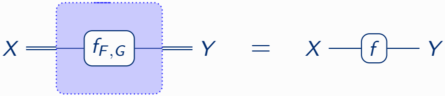




If f is a morphism in $\mathsf{C} \subseteq \mathsf{C}^{\otimes\infty}$, then



Infinite free model

Recall. From a symmetric monoidal theory (Σ, E), where Σ are generators of type $n \to m$ ($n, m \in \mathbb{N}$) and E is a set of equations between Σ-terms, one obtains $\mathbf{Free}(\Sigma, E)$.

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$\Rightarrow$ $\mathbf{Free}^{\infty}(\Sigma, E)$, with objects $\mathbb{N} \cup \{\infty\}$.

Proposition

$\mathbf{Free}^{\infty}(\Sigma, E) \simeq \mathbf{Free}(\Sigma, E)^{\otimes\infty}$.

Axiomatizing BinStoch

BinStoch $\underset{\text{full}}{\subseteq}$ **FinStoch** given by powers of 2 has complete axiomatisations $\mathbf{BinStoch} \cong \mathbf{Free}(\Sigma, E)$.[2,3]

$\Rightarrow \mathbf{BinStoch}^{\otimes\infty} \simeq \mathbf{Free}^{\infty}(\Sigma, E)$.

[2]Piedeleu, R., M. Torres-Ruiz, A. Silva and F. Zanasi, A complete axiomatisation of equivalence for discrete probabilistic programming, ESOP 2025.

[3]Di Giorgio, A., P. Sobociński and N. Voorneveld, Parametric Iteration in Resource Theories, CSL 2026.

Adding Infinite Tensor Products to FinStoch

Some preliminaries

1. By restricting to cancellative deletions: if $\mathsf{C} \to \mathsf{D}$ is faithful, with D having ITPs, then $\mathsf{C}^{\otimes\infty} \to \mathsf{D}$ is faithful.

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2. By restricting to **FinSet**, the pro-completion is given by Stone spaces (compact Hausdorff topology with a basis of clopens) and continuous maps.
   $\Rightarrow$ Objects should be given by Stone spaces.

Locally constant Markov kernels

Given two Stone spaces X and Y, a locally constant Markov kernel X to Y is given by

$$\begin{array}{rccc} f\colon & \mathrm{Clopen}\,(Y) \times X & \to & [0,1] \\ & (U, x) & \mapsto & f(U \,|\, x) \end{array}$$

such that $f(U \,|\, -)\colon X \to [0,1]$ is locally constant and $f(- \,|\, x)\colon \mathrm{Clopen}\,(Y) \to [0,1]$ is a (finitely-additive) probability measure.

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$\mathbf{CantorStoch}_{\mathrm{lc}} \underset{\text{full}}{\subseteq} \mathbf{StoneStoch}_{\mathrm{lc}}$ given by finite powers of 2 and the Cantor space $2^{\mathbb{N}}$.

Main theorem

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Corollary

The symmetric monoidal theory (Σ, E) for **BinStoch** satisfies $\mathbf{Free}^{\infty}(\Sigma, E) \cong \mathbf{CantorStoch}_{\mathsf{lc}}$.

Recovering probability measures

All probability measures are locally constant Markov kernels:

$$\mathbf{BorelStoch}(I, \mathbb{R}) \cong \mathbf{BorelStoch}(I, 2^{\mathbb{N}}) \cong \mathbf{CantorStoch}_{\mathsf{lc}}(I, 2^{\mathbb{N}}),$$

where the second one follows by taking the finite approximation families $(I \to 2^F)_{F \subseteq \mathbb{N}}$.

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The same holds by replacing I with any finite set.

The inclusion of $2^{\mathbb{N}} \hookrightarrow [0, 1]$ identifying the Cantor set can be used to define a Markov kernel $2^{\mathbb{N}} \to 2$ that is not locally constant.

Future work

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2. What happens when we consider other categories, such as **Gauss** and **FinSetMulti**?
3. What can be said for partial Markov categories, or distributive Markov categories with tape diagrams?

THANK YOU FOR YOUR ATTENTION

Locally constant Markov kernels on the Cantor space

Let us consider $X = Y = 2^{\mathbb{N}}$.

Clopen sets can be described as finite sequences $(a_1, \ldots, a_k)$ $(U = \{(x_n) \in 2^{\mathbb{N}} \mid x_i = a_i \ \forall i \leq k\})$.

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Then locally constant Markov kernels $2^{\mathbb{N}} \to 2^{\mathbb{N}}$ satisfy

$$f((a_1, \dots, a_k) \mid (b_1, b_2, \dots, b_n, \dots)) = f((a_1, \dots, a_k) \mid (b_1, b_2, \dots, b_n))$$

for some n.