From Bisimilarity to ε-Bisimilarity

Ana Sokolova

PARIS LODRON UNIVERSITY SALZBURG

Why formal methods?

Therac-25

Therac-25 is a computer-controlled radiation therapy machine produced by Atomic Energy of Canada Limited (AECL) in 2 after the Therac-6 and Therac-20 units (the earlier units had been produced in partnership with Compagnie générale de radiologie (CGR) of France).[1]

The Therac-25 was involved in at least six accidents between 1985 and 1987, in which some patients were given massive overdoses of radiation.[2]:425 Because of concurrent programming errors (also known as race conditions), it sometimes gave its patients radiation doses that were hundreds of times greater than normal, resulting in death or serious injury.[3] These accidents highlighted the dangers of software control of safety-critical systems.

The Therac-25 has become a standard case study in health informatics, software engineering, and computer ethics. It highlights the dangers of engineer overconfidence[2]:428 after the engineers dismissed end-user reports, leading to severe consequences.

History [edit]

The French company CGR manufactured the *Neptune* and *Sagittaire* linear accelerators.

In the early 1970s, CGR and the Canadian public company *Atomic Energy of Canada Limited* (AECL) collaborated on the construction of linear accelerators controlled by a DEC

Ariane-5

"Cluster (spacecraft)" redirects here. This article is about the failed Cluster launch. For the successful space mission, see Cluster II (spacecraft).

Ariane flight V88[1] was the failed maiden flight of the Arianespace Ariane 5 rocket, vehicle no. 501, on 4 June 1996. It carried the **Cluster** spacecraft, a constellation of four European Space Agency research satellites.

The launch ended in failure due to multiple errors in the software design: dead code, intended only for Ariane 4, with inadequate protection against integer overflow led to an exception handled inappropriately, halting the whole otherwise unaffected inertial navigation system. This caused the rocket to veer off its flight path 37 seconds after launch, beginning to disintegrate under high aerodynamic forces, and finally self-destructing via its automated flight termination system. The failure has become known as one of the most infamous and expensive software bugs in history.[2] The failure resulted in a loss of more than US$370 million.[3]

Launch failure [edit]

Cluster

Mission type	Magnetospheric
Operator	ESA
Spacecraft properties	
Launch mass	1,200 kilograms (2,600 lb)
Start of mission	
Launch date	12:34:06, 4 June 1996 (UTC)
Rocket	Ariane 5G
Launch site	Kourou ELA-3
End of mission	
Disposal	launch failure
Destroyed	4 June 1996

esa

Intel Pentium

"Pentium bug" redirects here. For the 1997 bug affecting Pentium processors, see Pentium F00F bug.

The **Pentium FDIV bug** is a hardware bug affecting the floating-point unit (FPU) of the early Intel Pentium processors. Because of the bug, the processor would return incorrect binary floating point results when dividing certain pairs of high-precision numbers. The bug was discovered in 1994 by Thomas R. Nicely, a professor of mathematics at Lynchburg College.[1] Missing values in a lookup table used by the FPU's floating-point division algorithm led to calculations acquiring small errors. While these errors would in most use-cases only occur rarely and result in small deviations from the correct output values, in certain circumstances the errors can occur frequently and lead to more significant deviations.[2]

The severity of the FDIV bug is debated. Though rarely encountered by most users (*Byte*

66 MHz Intel Pentium

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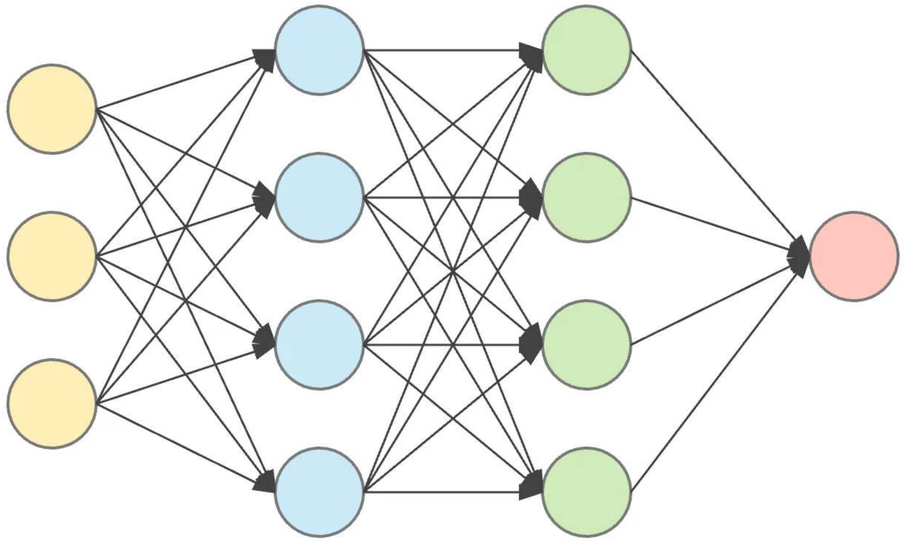

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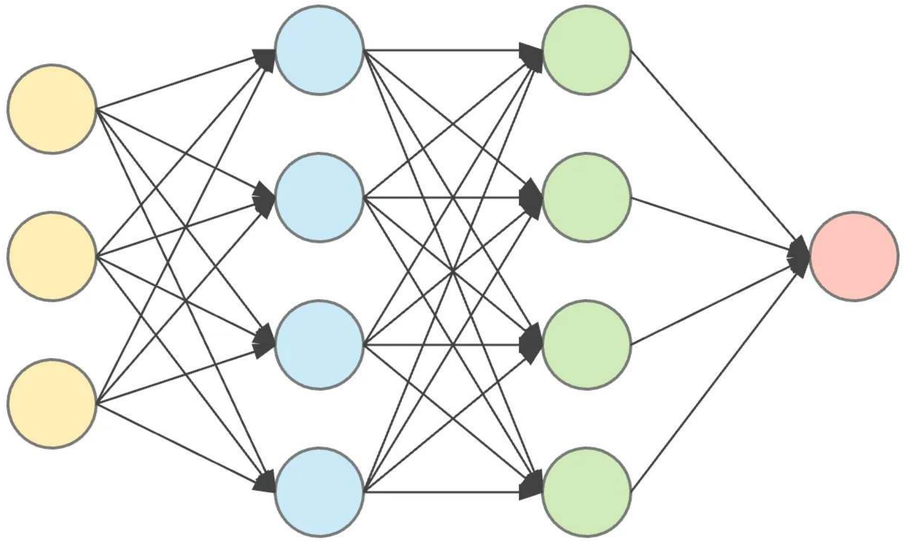

Do they do what we expect them to do ?

How do formal methods help?

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In my work: with behavioural equivalences, like bisimilarity or trace semantics

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probabilistic programs

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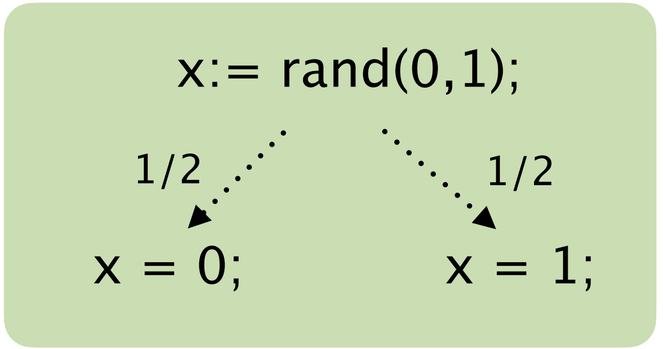

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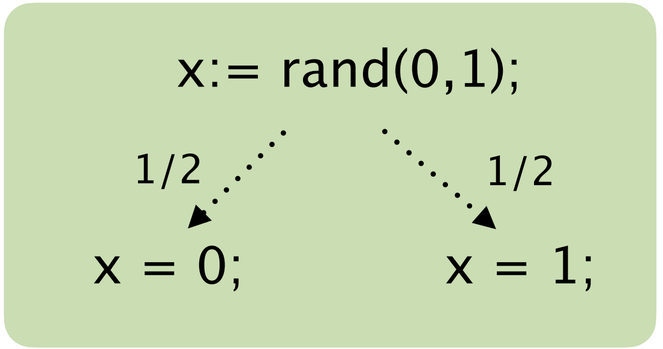

verifying **equivalence** of programs

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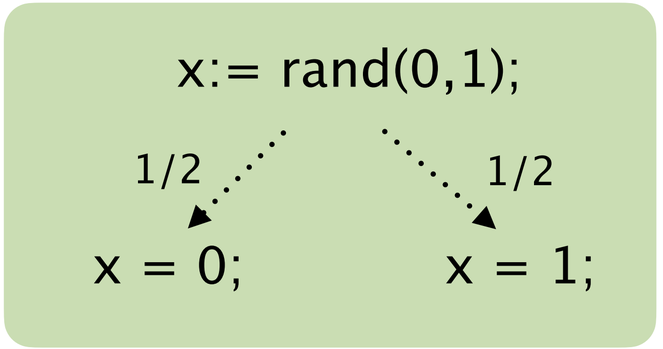

verifying **equivalence** of programs

P ≡ Optimised(P)

Source ≡ Compiled

Implementation ≡ Specification

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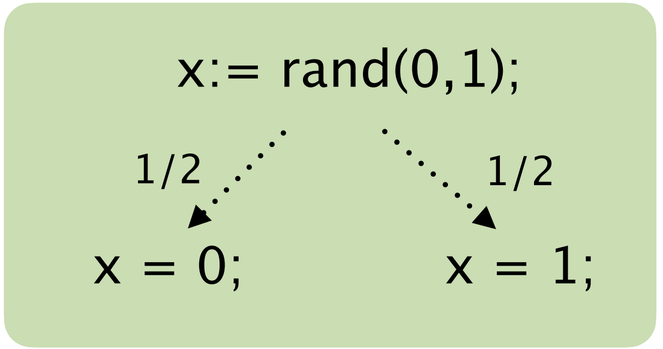

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developing **proof techniques**

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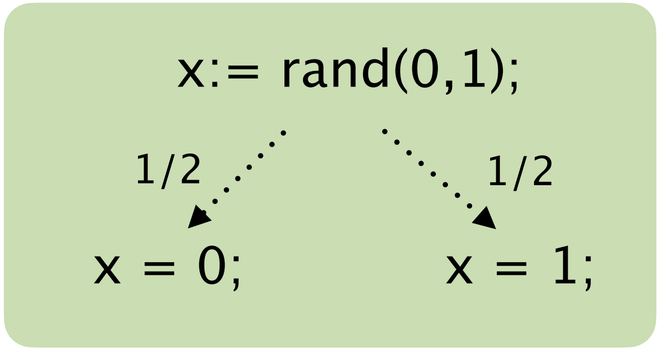

verifying **equivalence** of programs

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developing **proof techniques**

e.g. by axiomatisations / up-to techniques

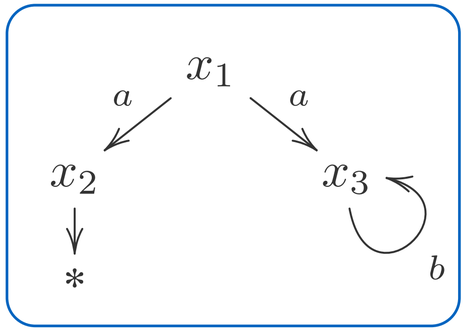
Coalgebras and Bisimilarity

Some coalgebras

Some coalgebras

NFA

$X \to 2 \times (\mathcal{P}X)^A$



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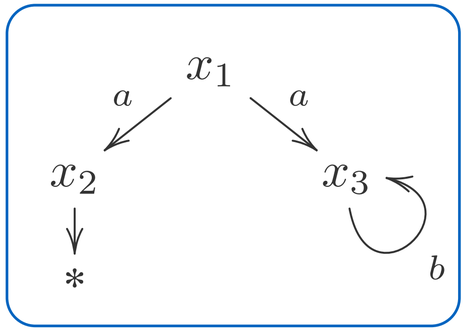
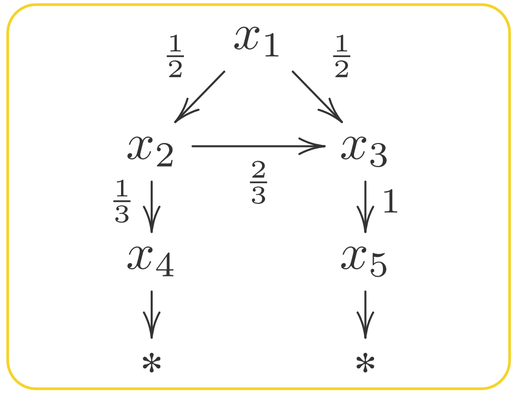


MC

$X \to \mathcal{D}X + 1$



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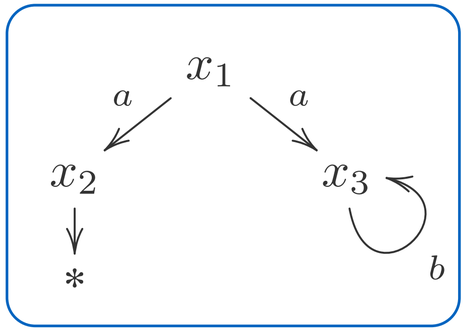

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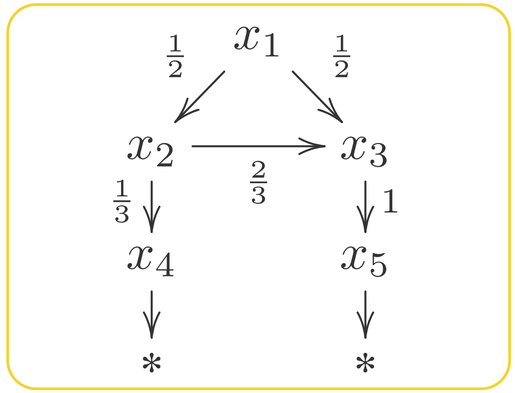

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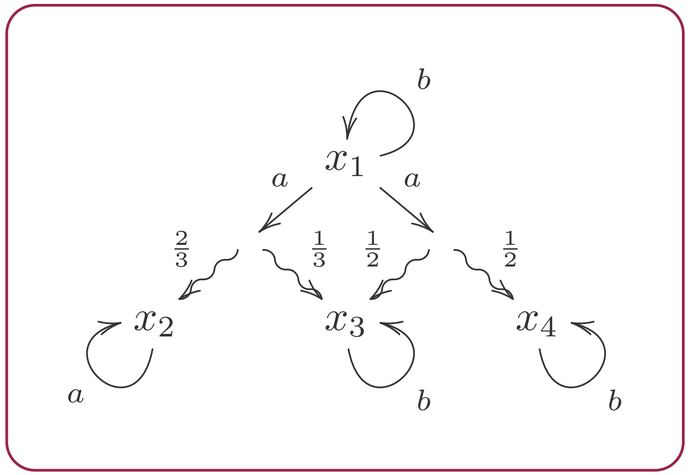

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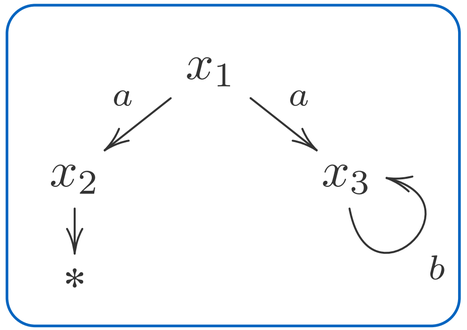

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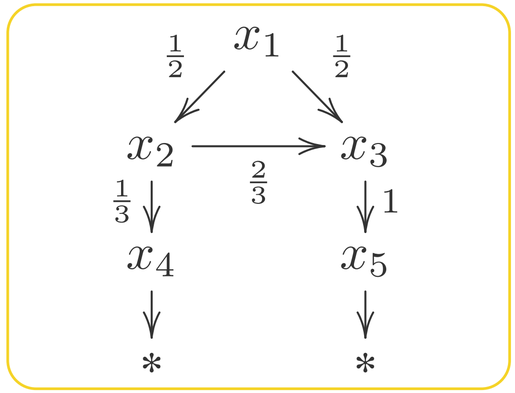

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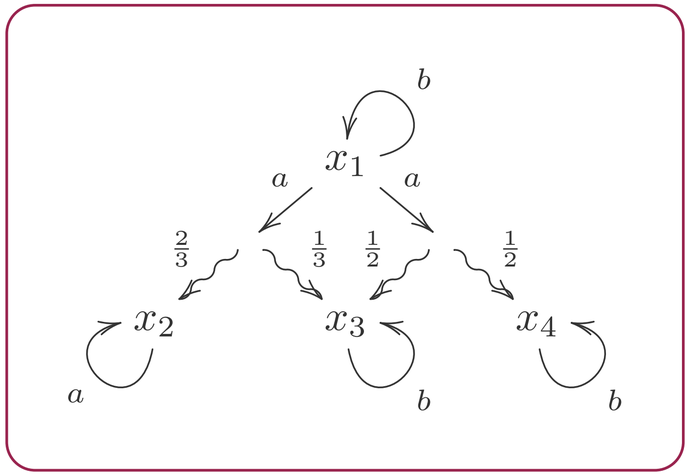

Various transitions systems / automata are coalgebras

Coalgebra

Uniform framework for dynamic transition systems, based on category theory.

A coalgebra is a generic transition system:

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$$X \longrightarrow FX$$

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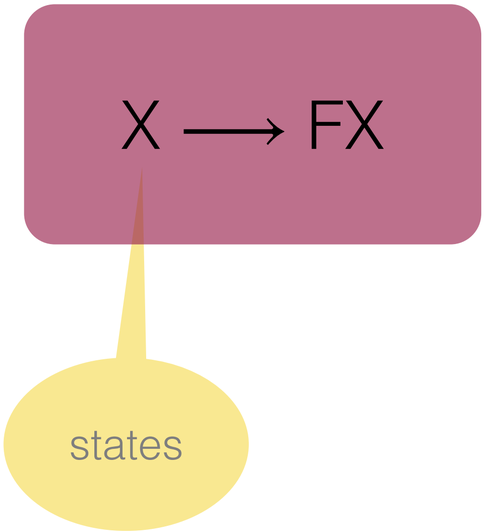

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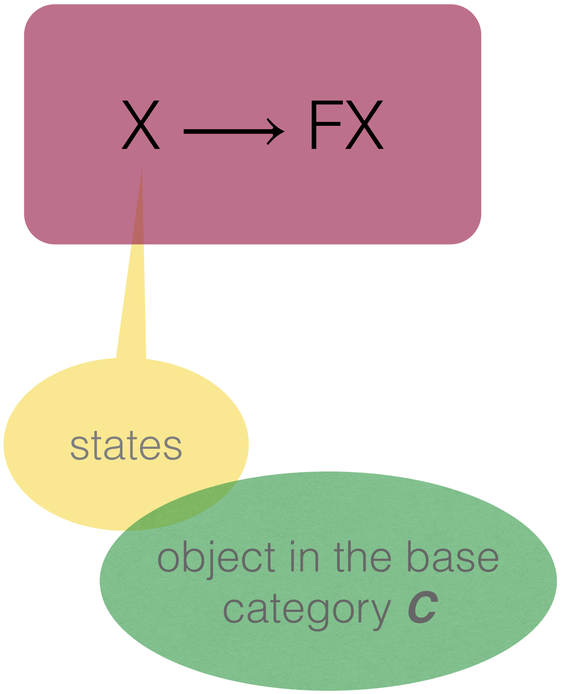

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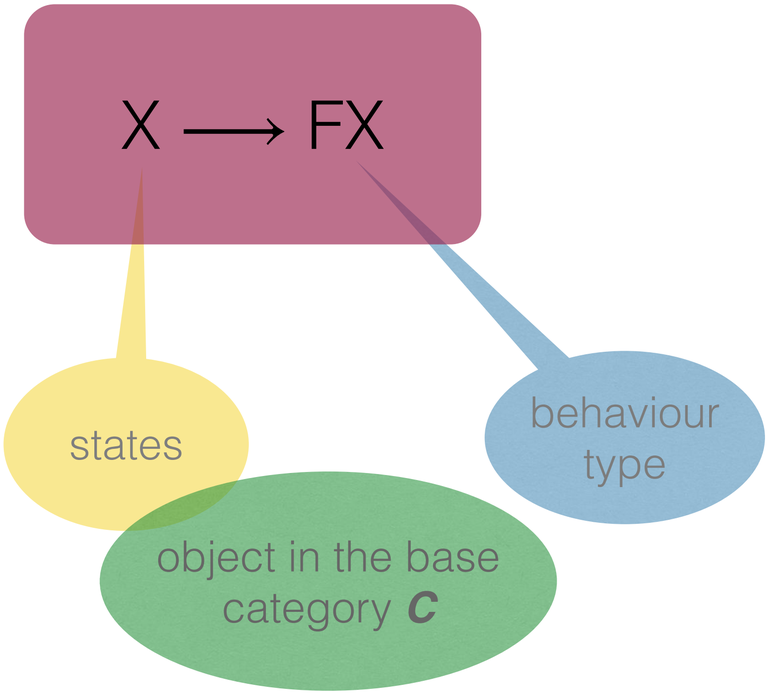

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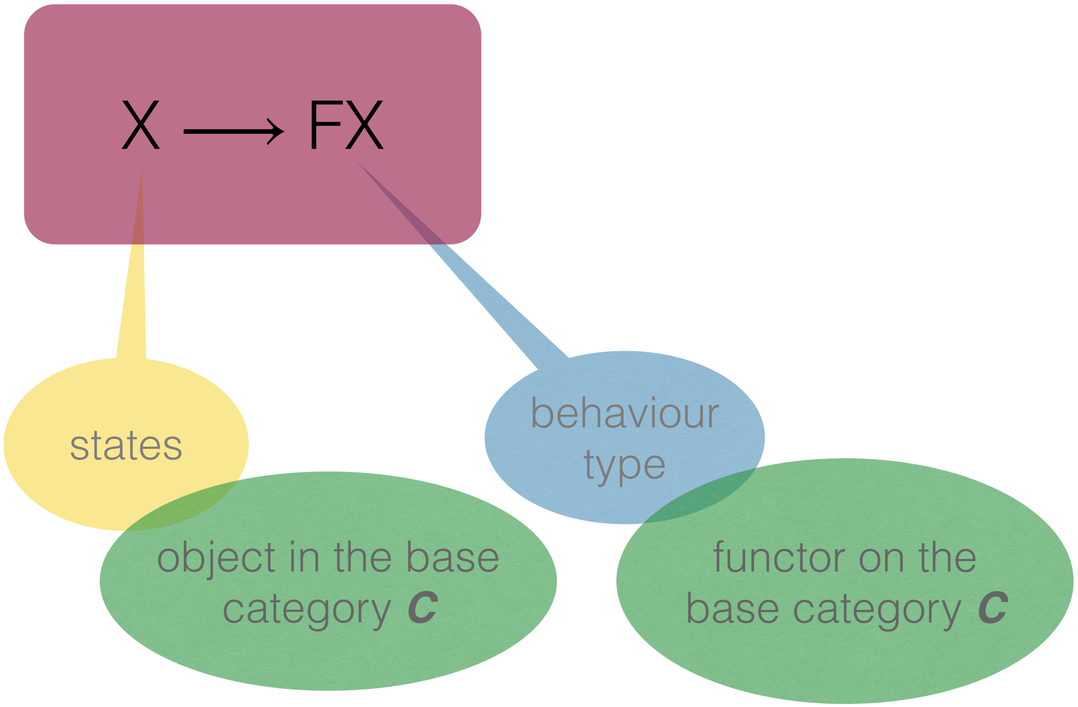

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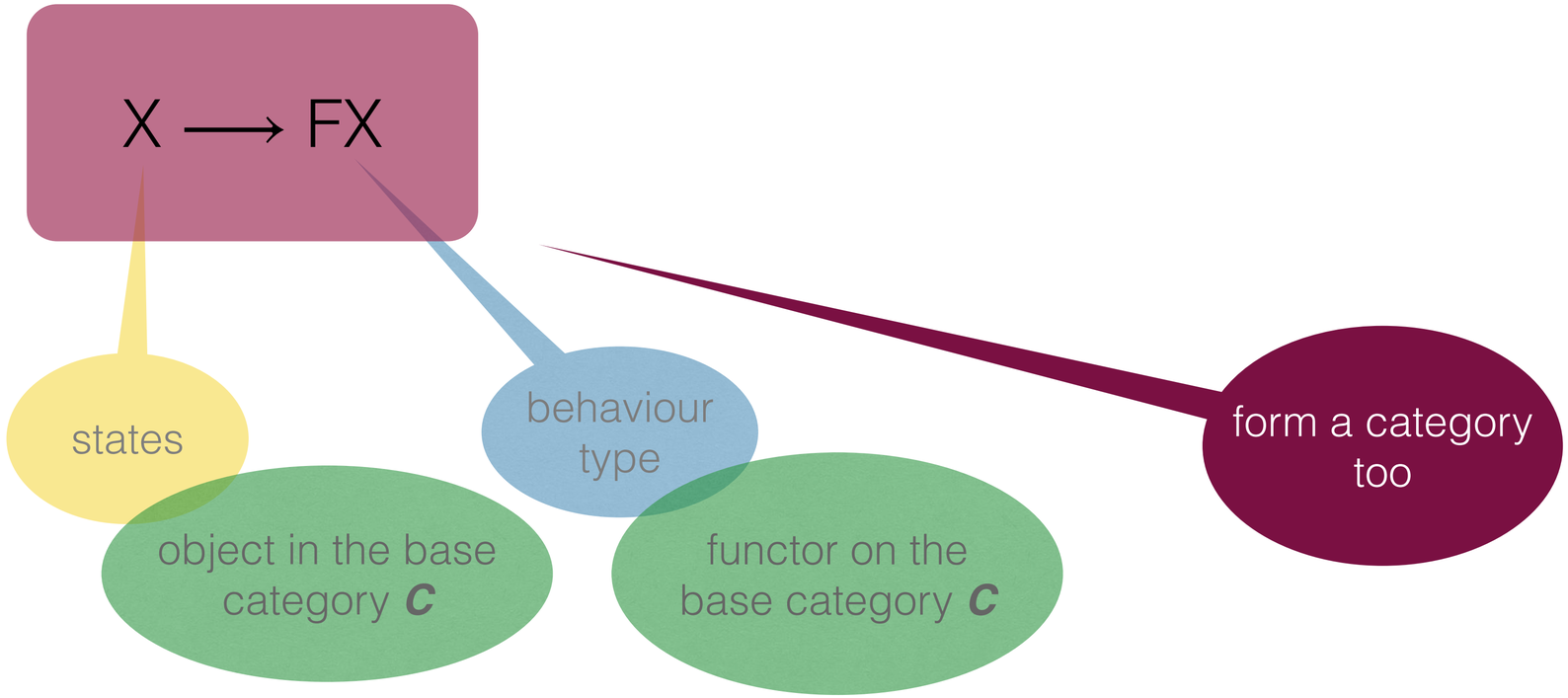

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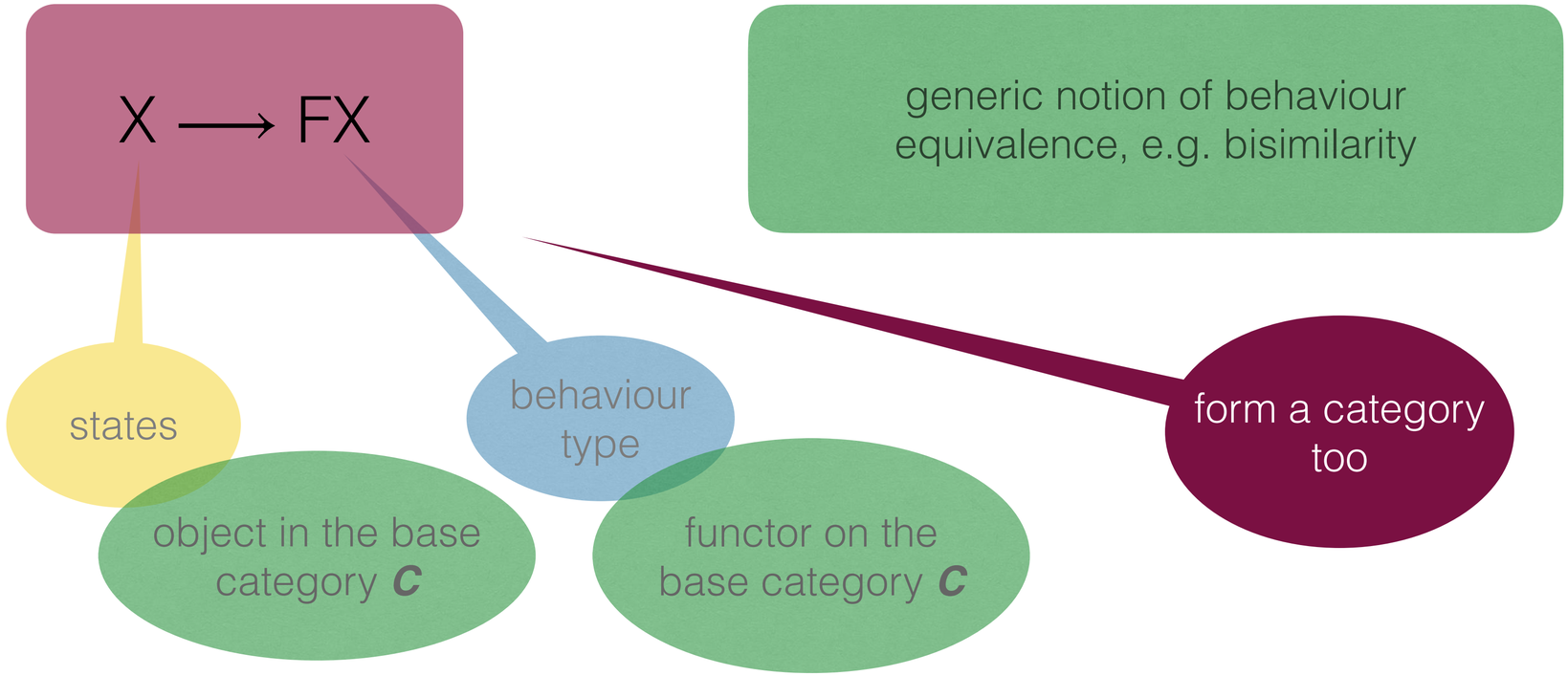

Bisimilarity of Labelled Transition Systems

$S \to (\mathcal{P}\, S)^A$

Bisimilarity of Labelled Transition Systems

An LTS is a map $S \to (\mathcal{P}\, S)^A$

with a set of states S and a set of actions A.

A binary relation R between the states of two LTS is a bisimulation

iff whenever two states are related, each transition from one of them

has a matching transition from the other and the target states are related again.

Two states are bisimilar iff they are related by a bisimulation.

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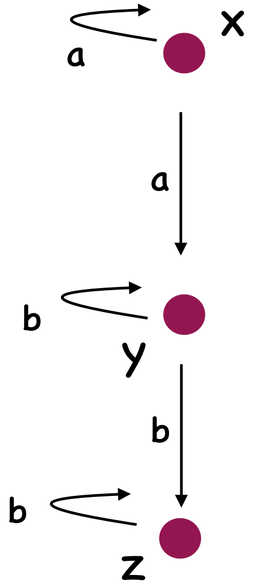

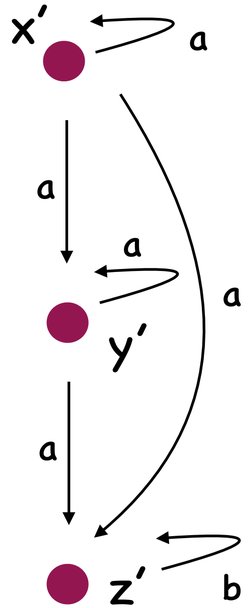

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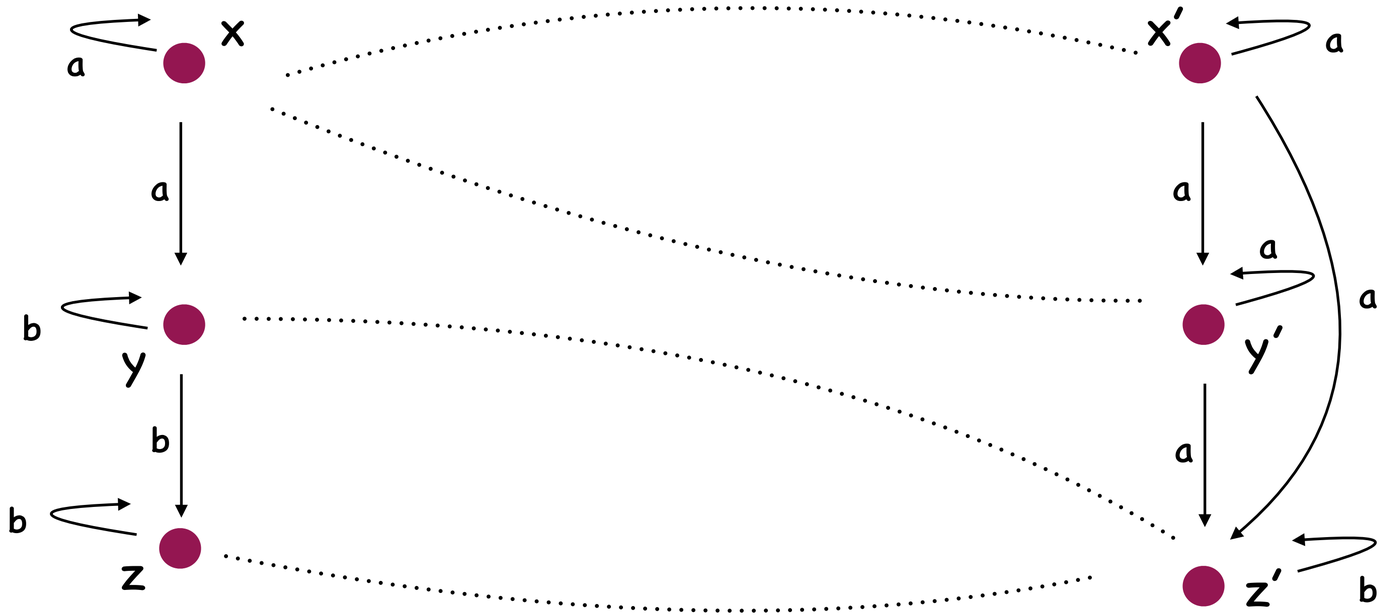

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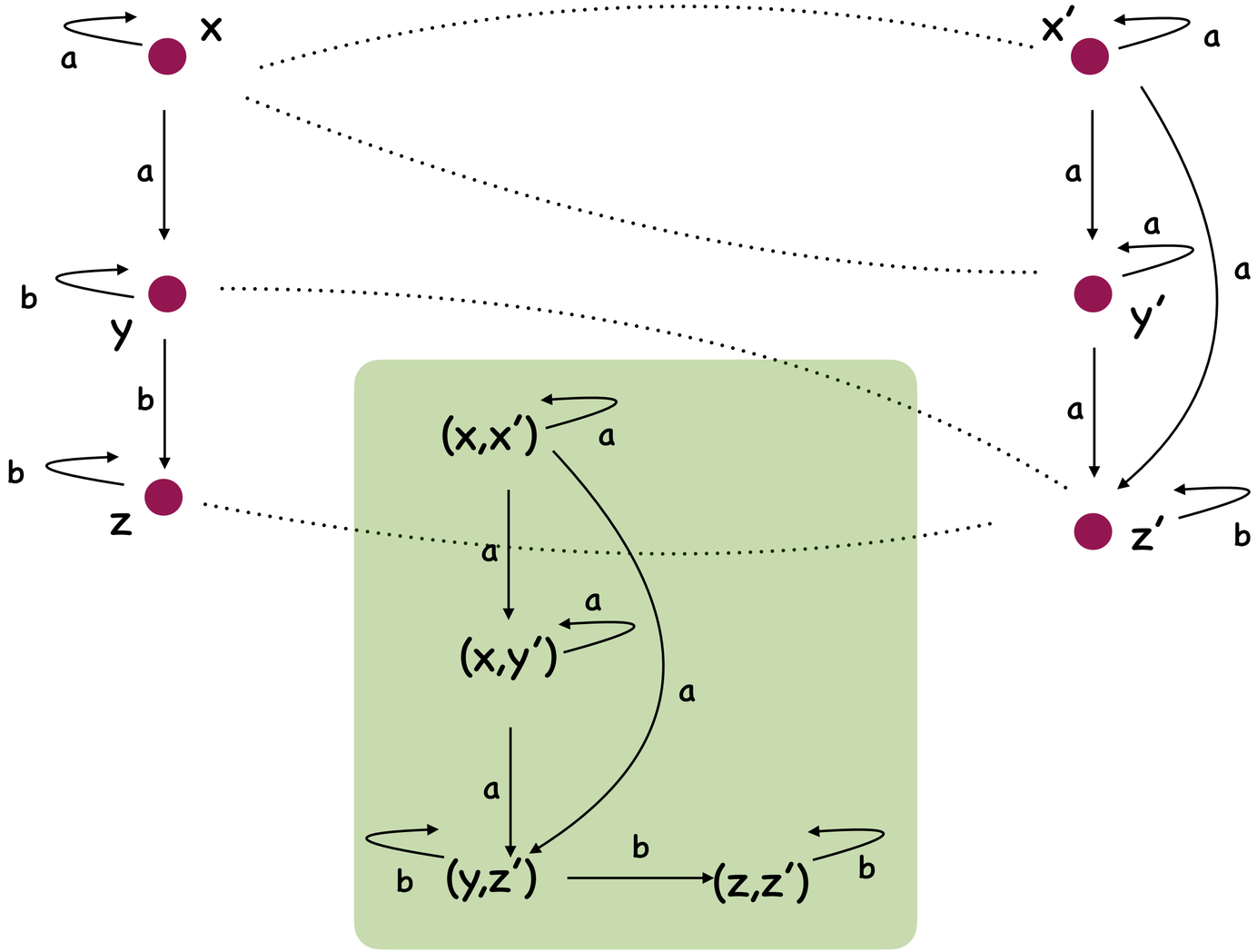

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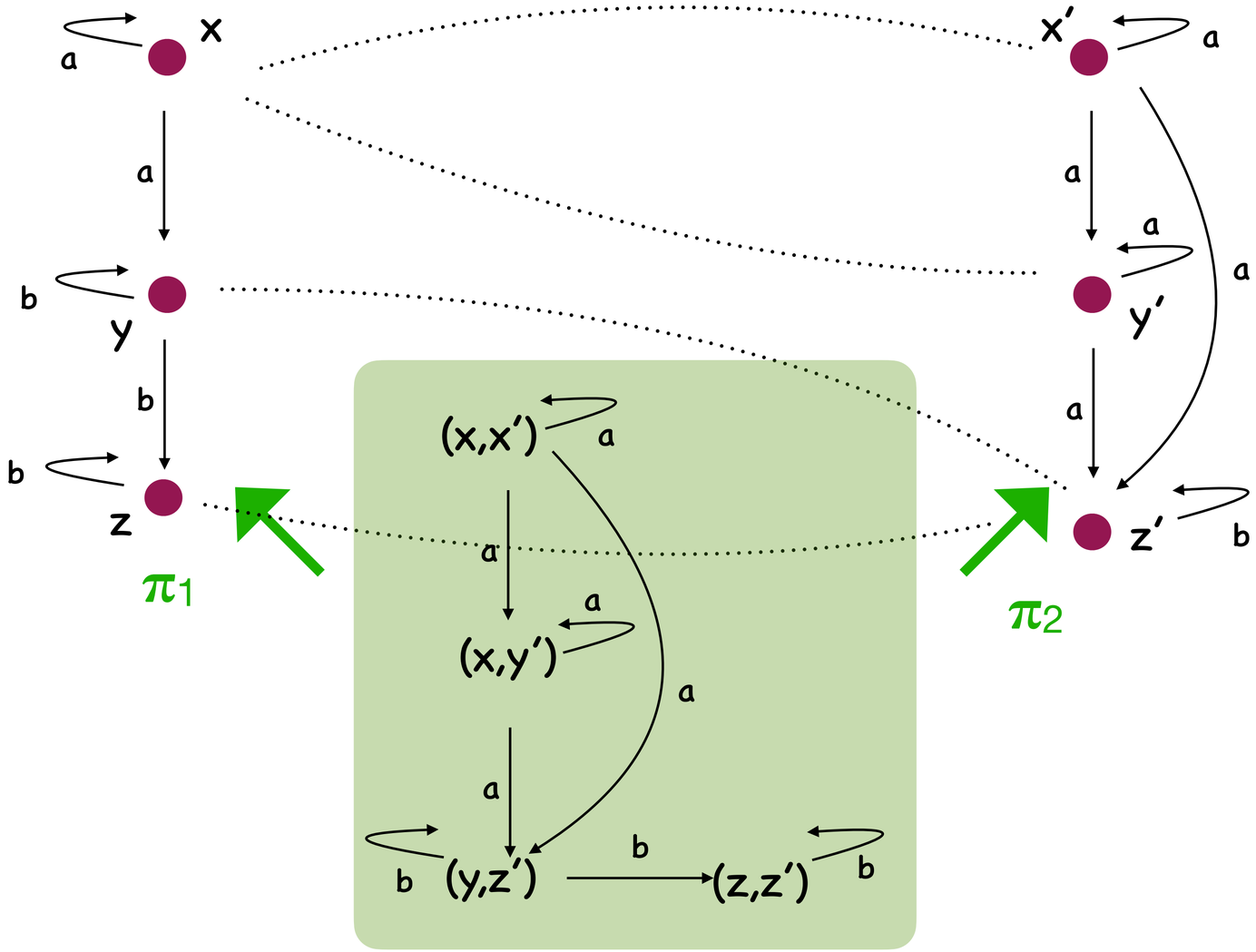

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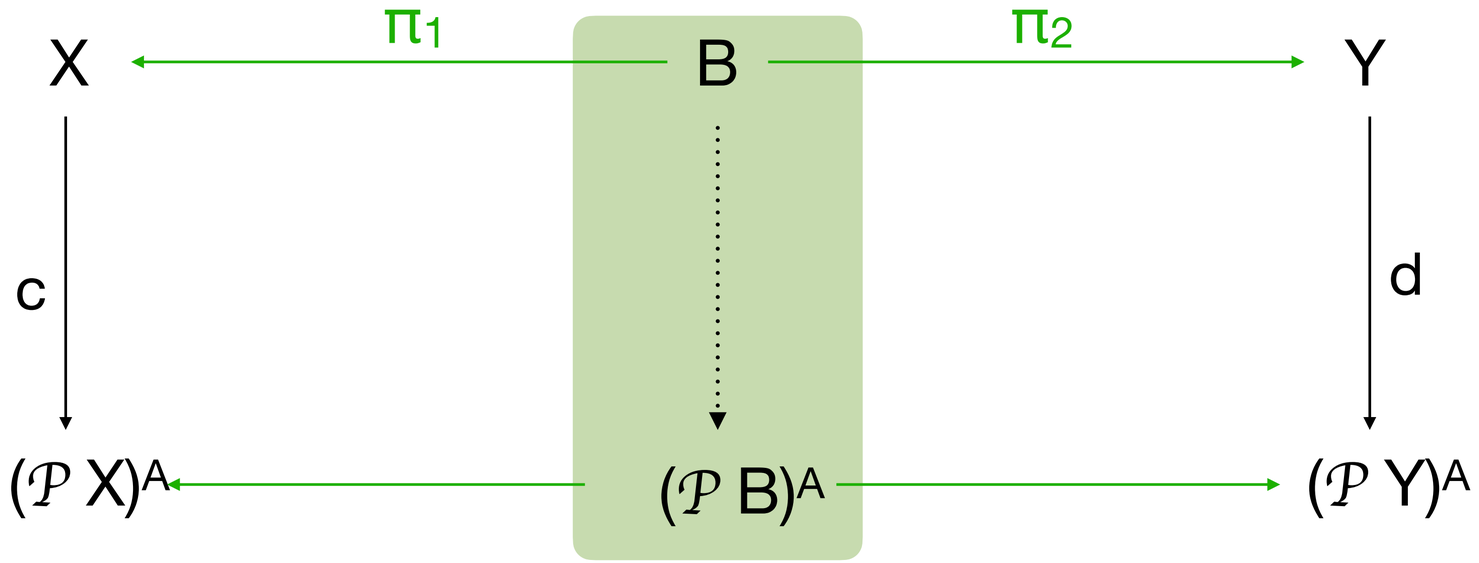

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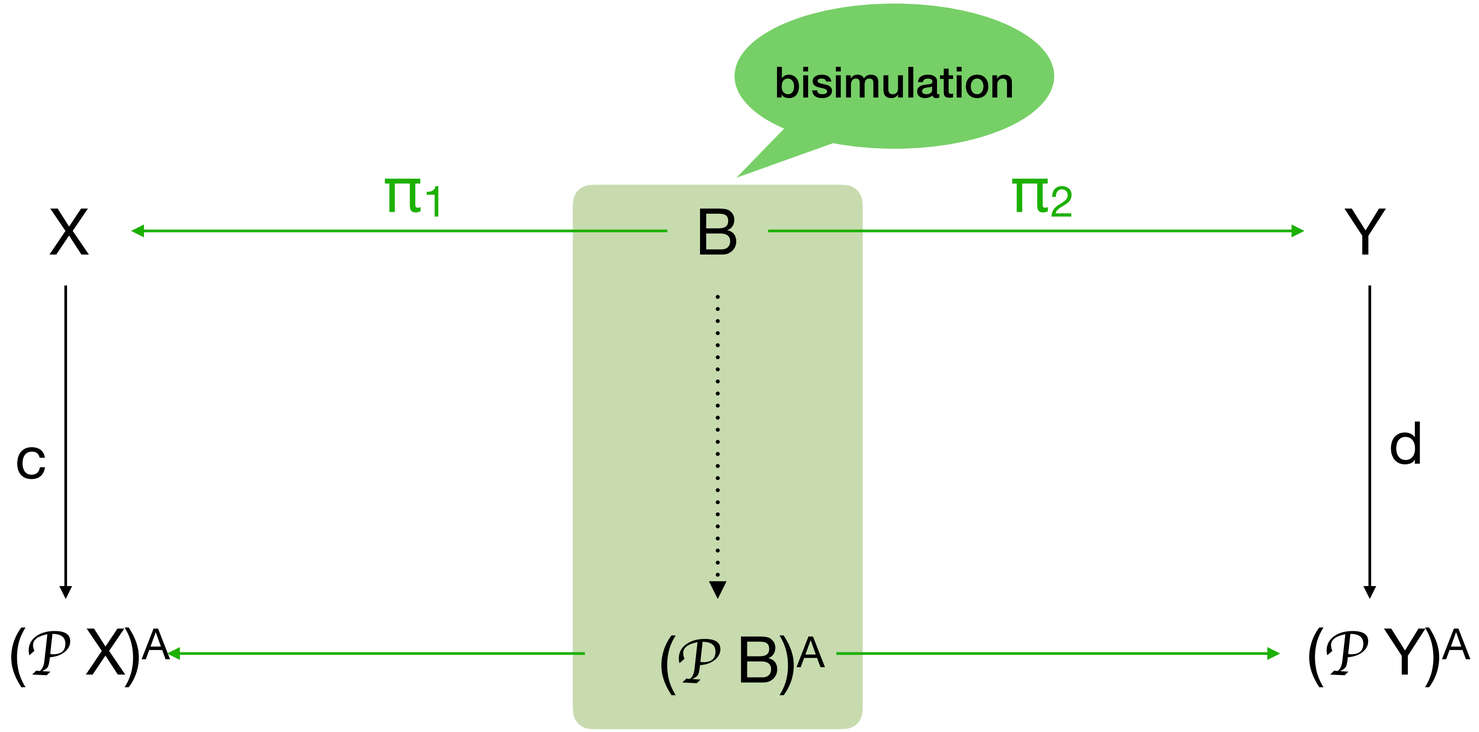
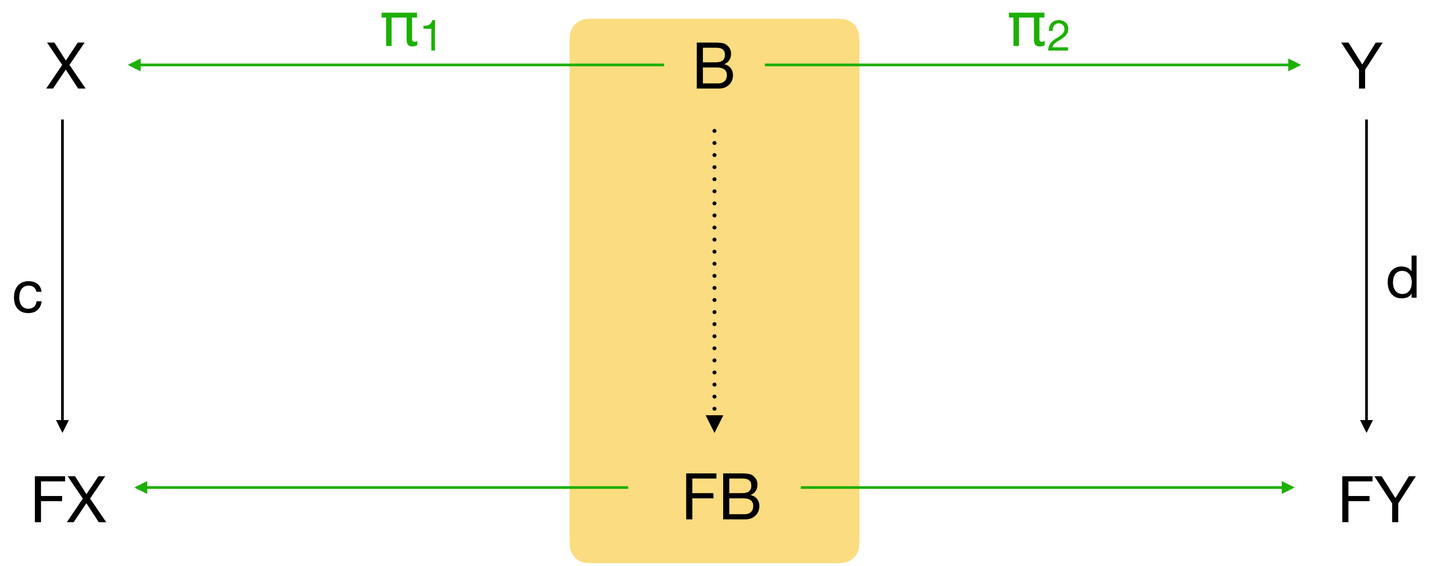


Bisimilarity of F-Coalgebras

S → FS



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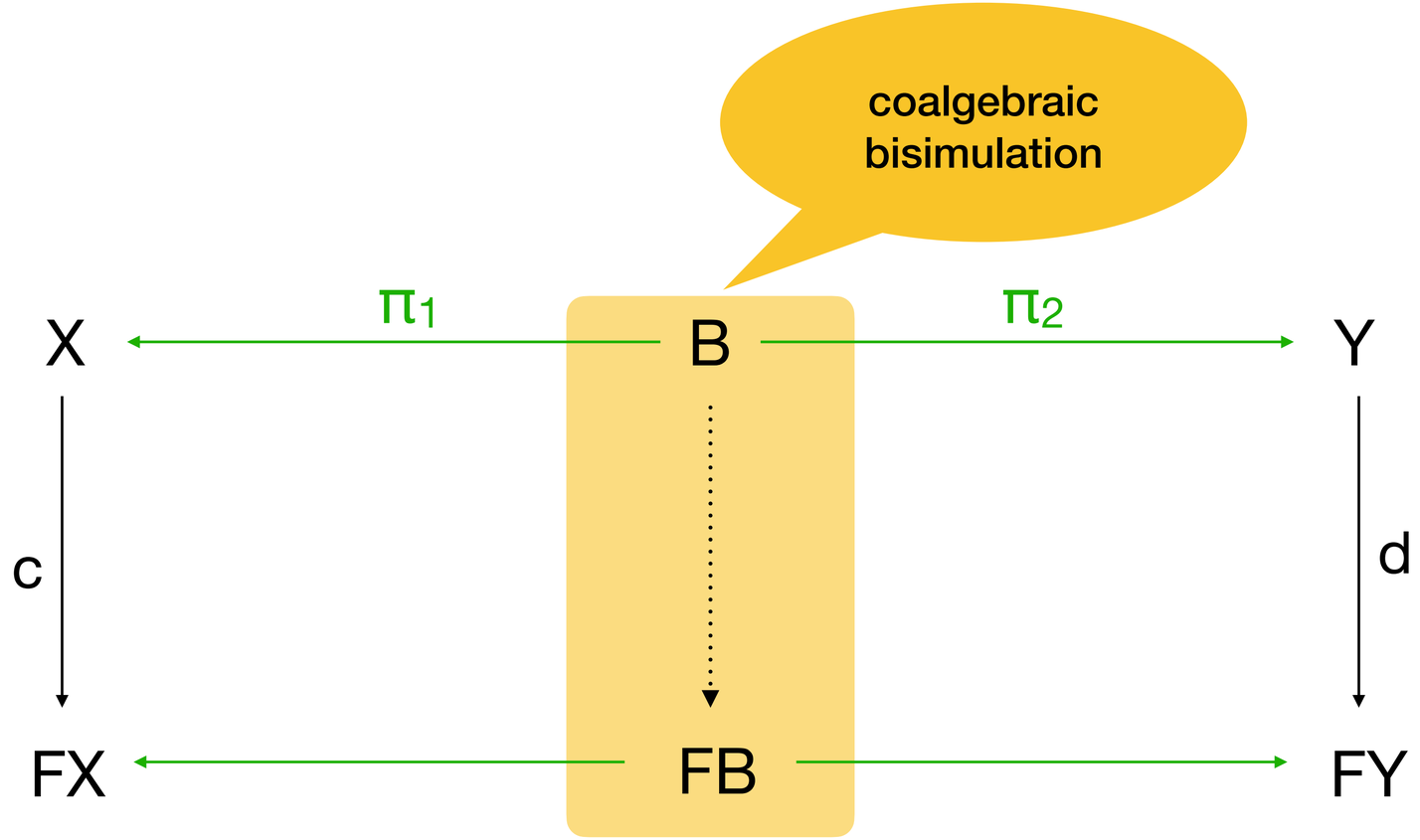

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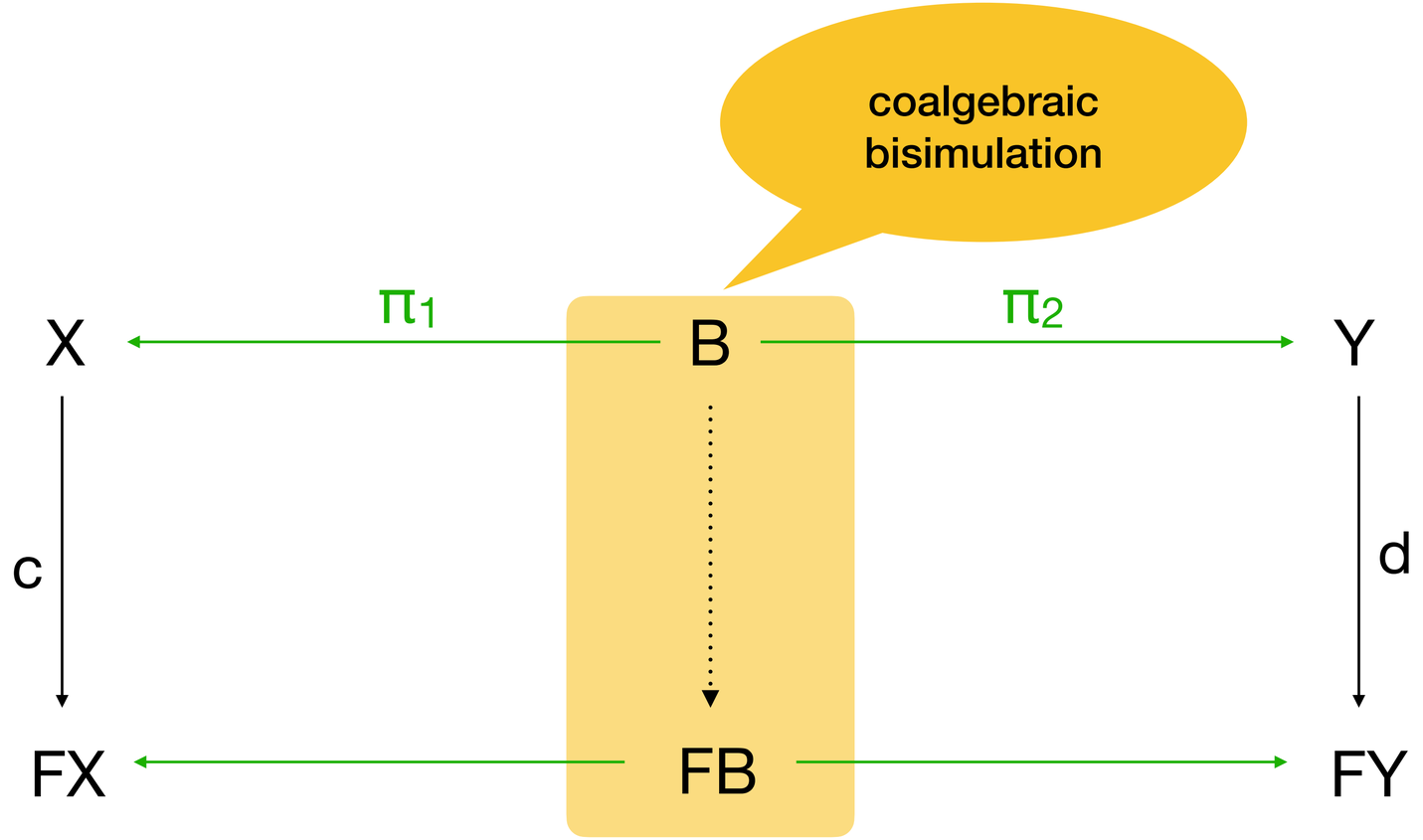

S→ FS

Aczel & Mendler 1989

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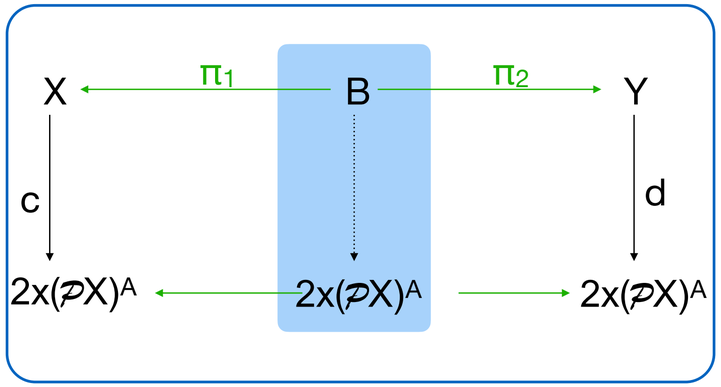

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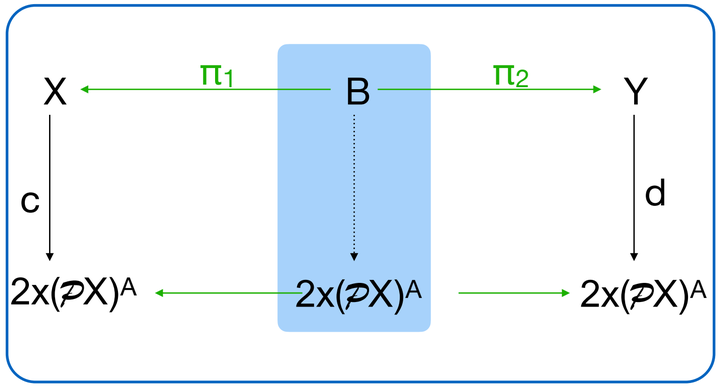

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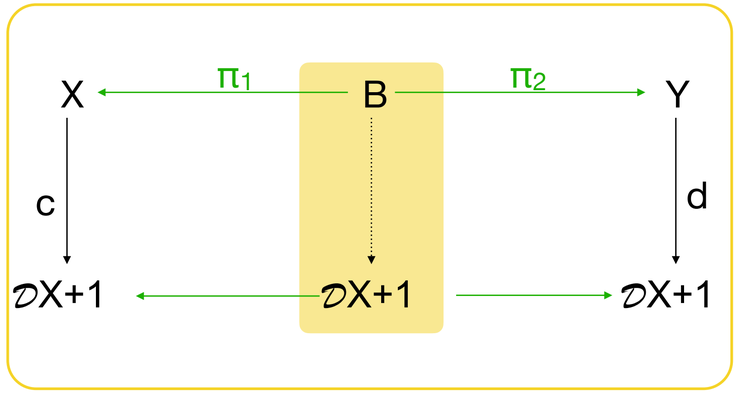

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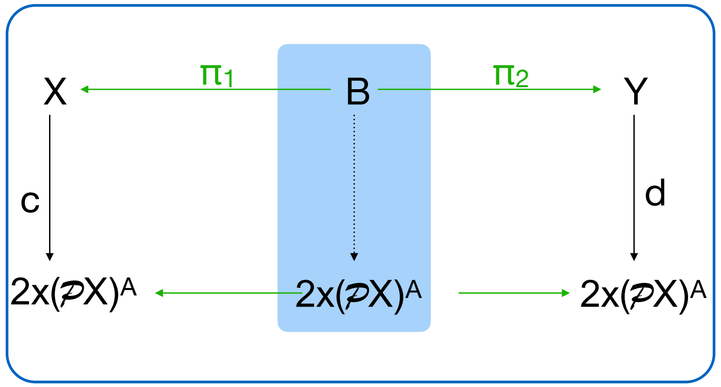
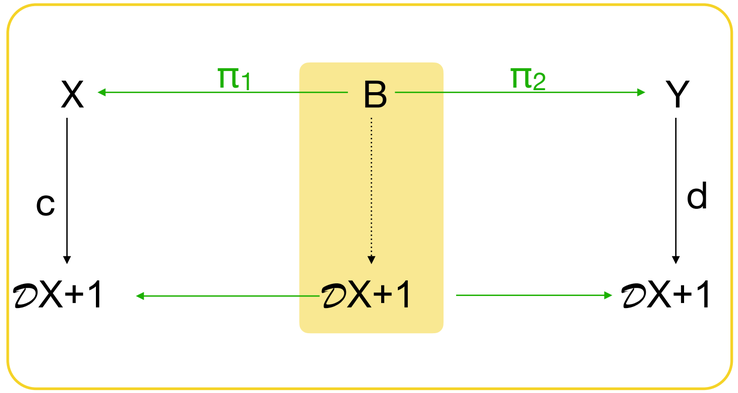


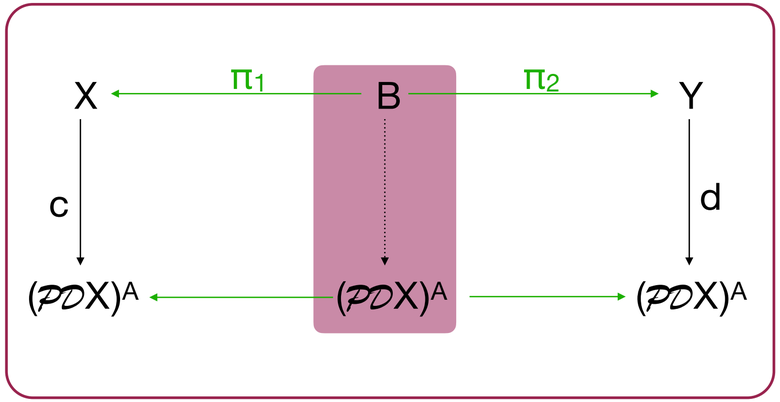
MC

$X \to \mathcal{D}X + 1$



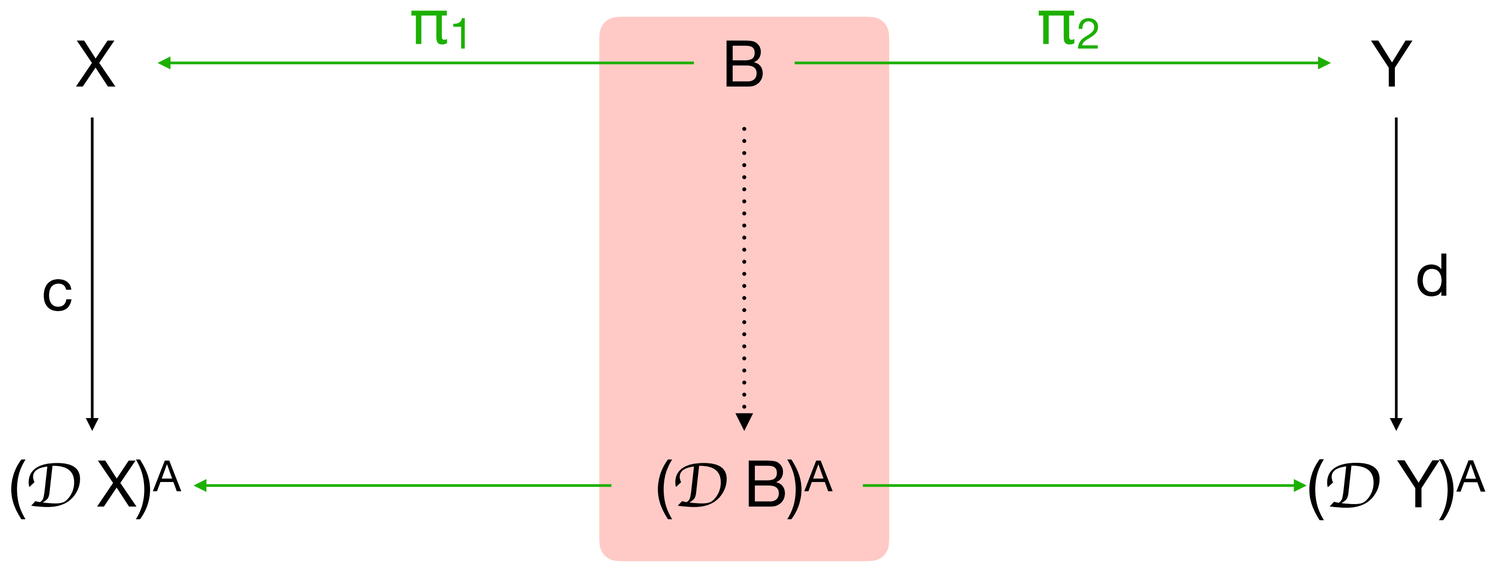
PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



Bisimilarity of Probabilistic Transition Systems

$S \to (\mathcal{D}\, S)^A$



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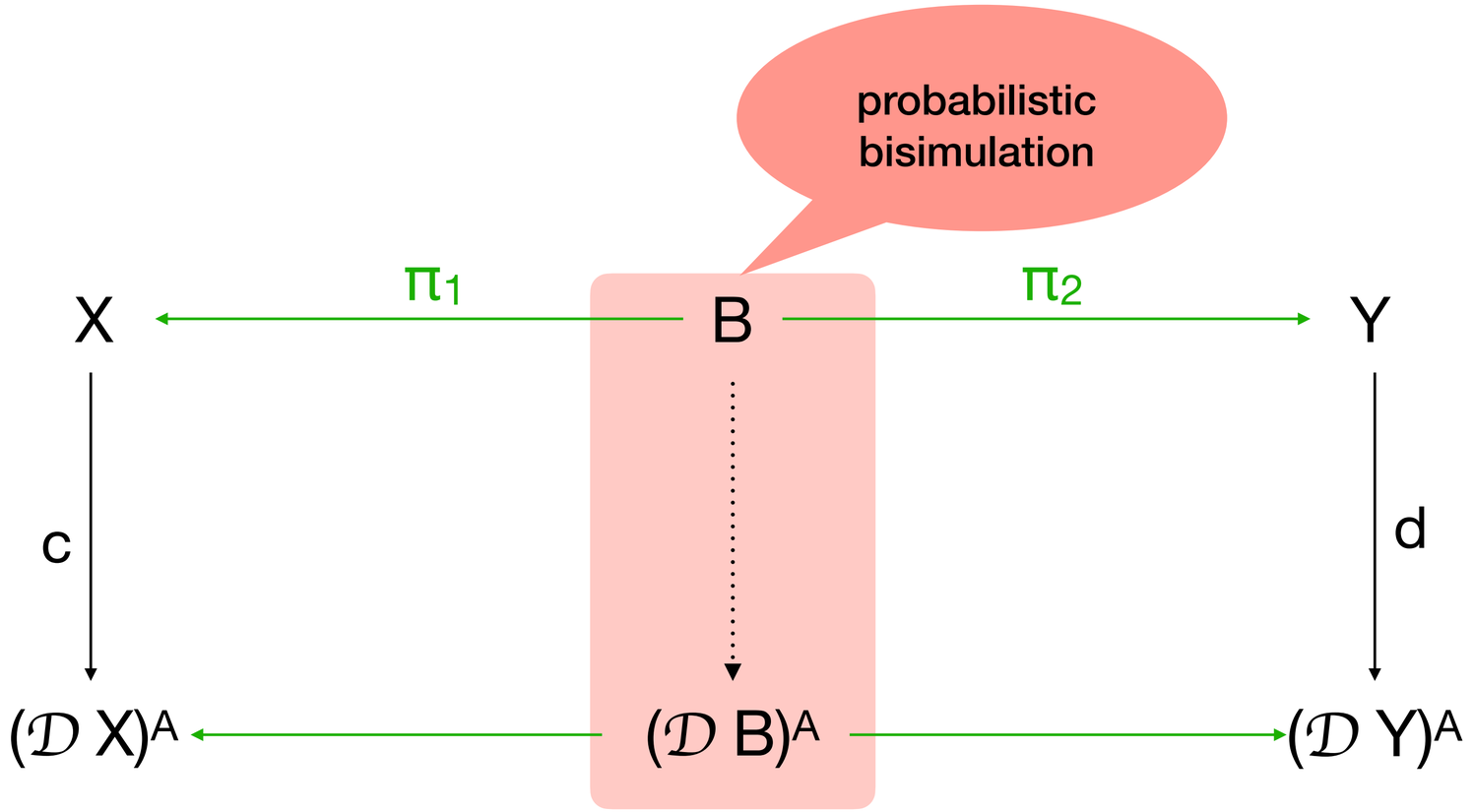

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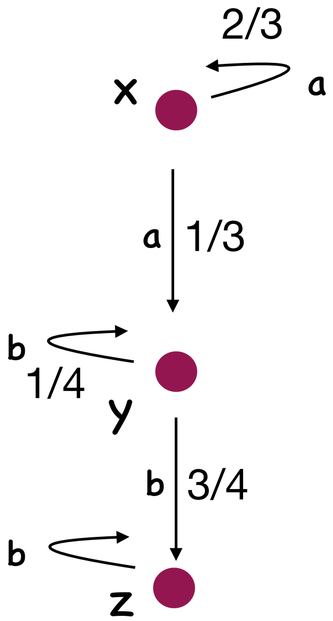

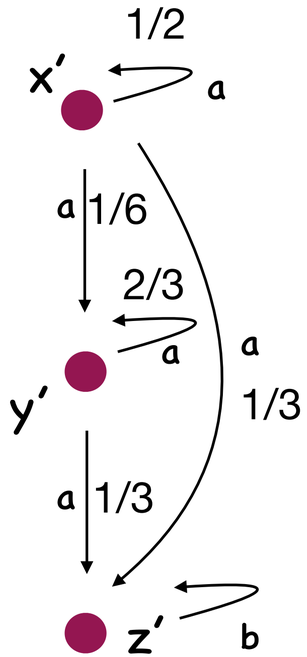

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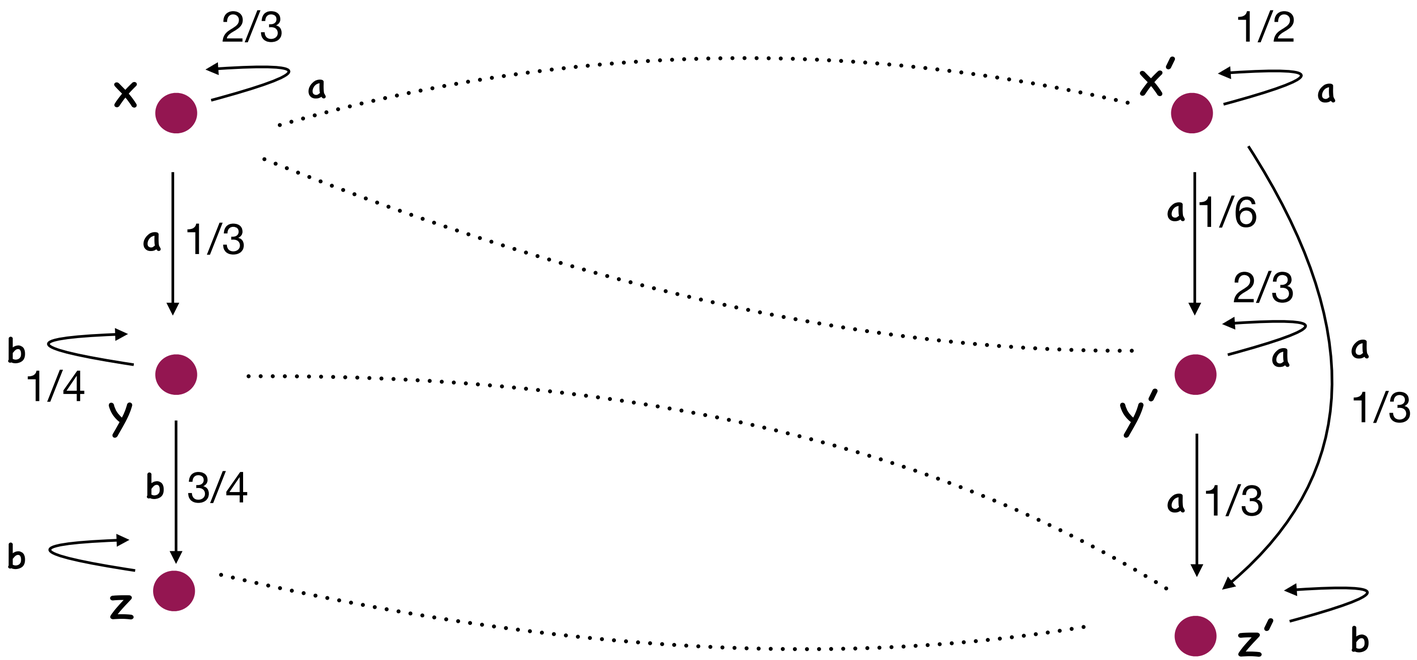

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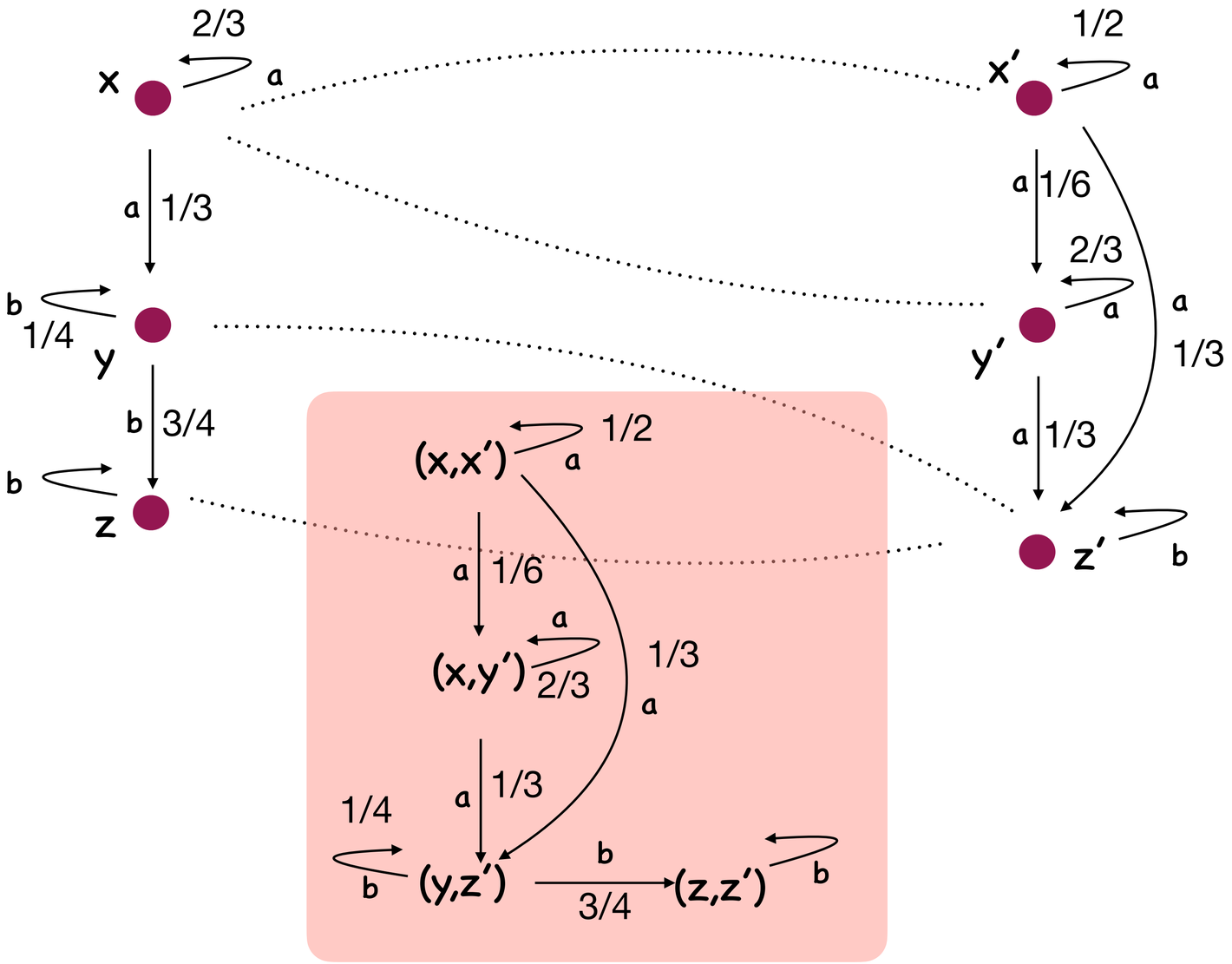

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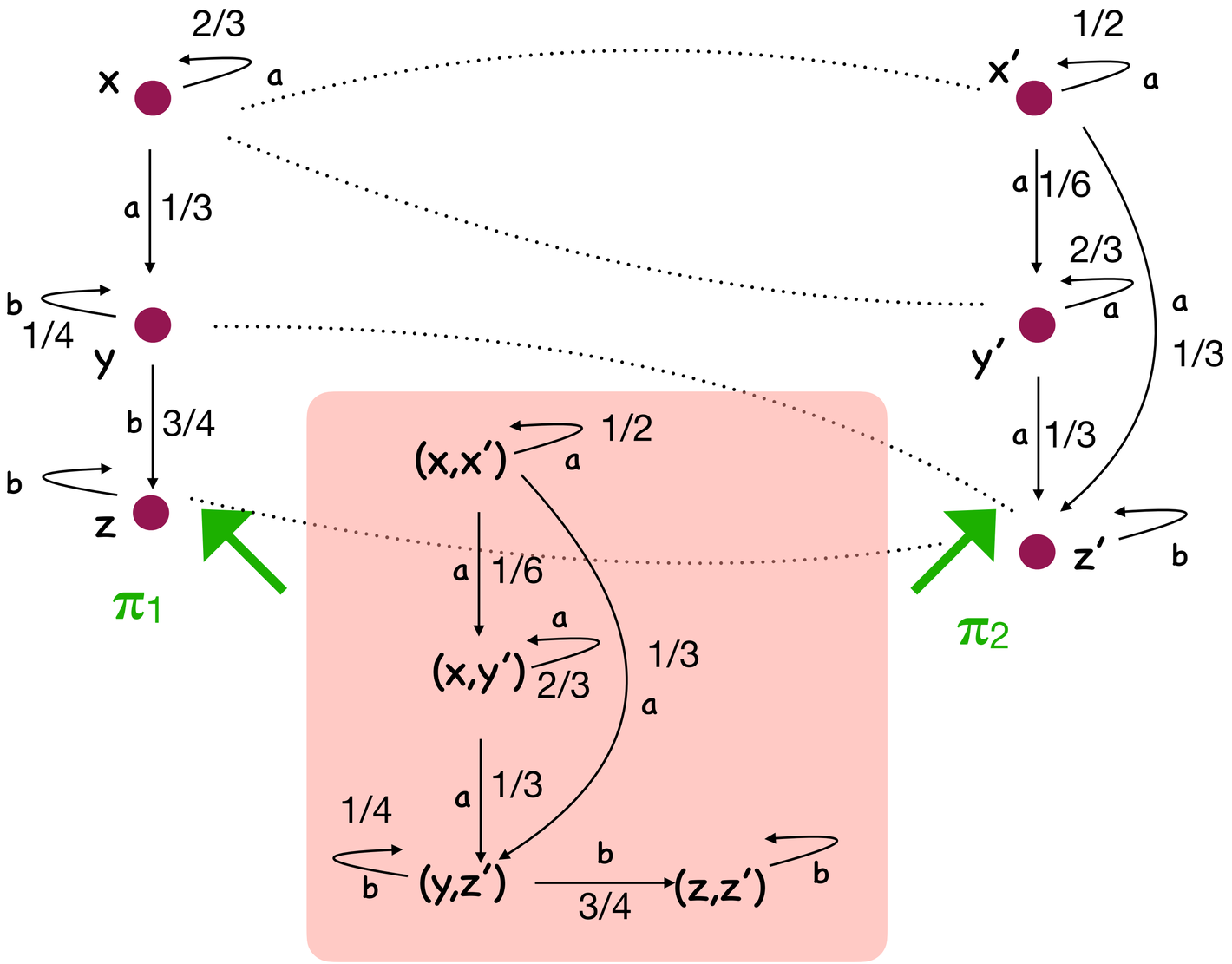

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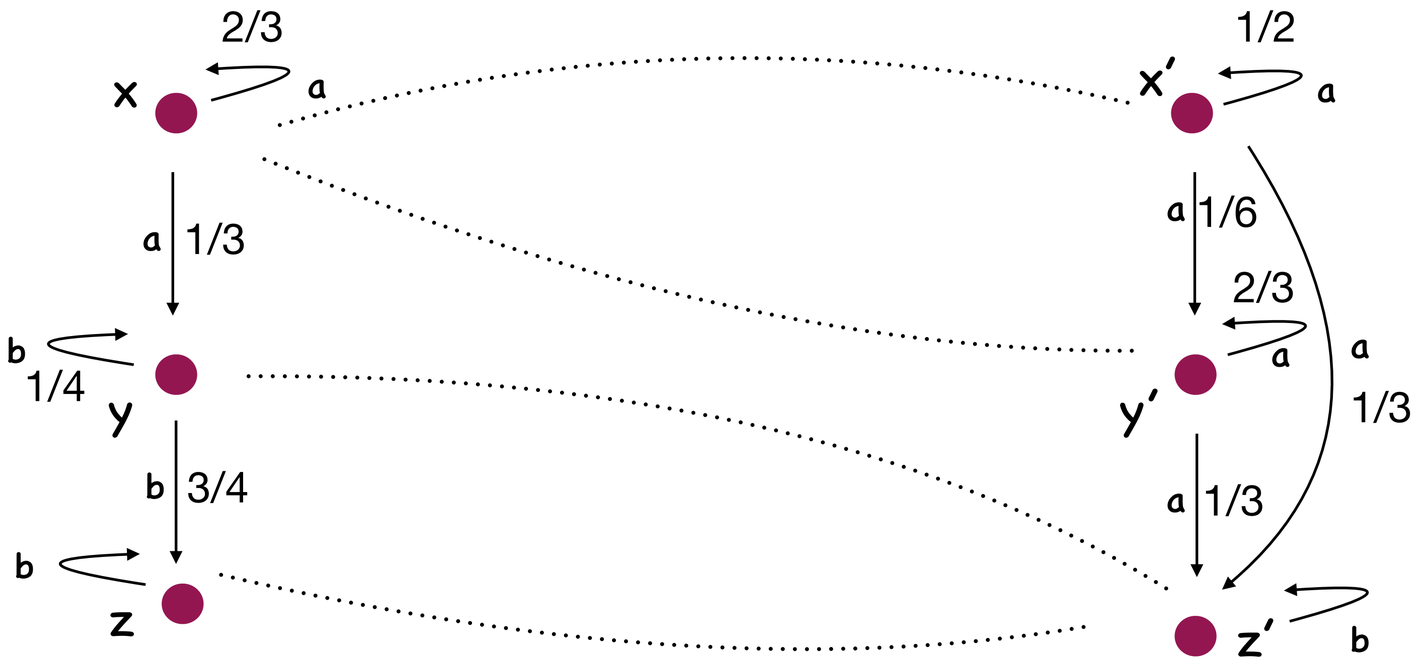

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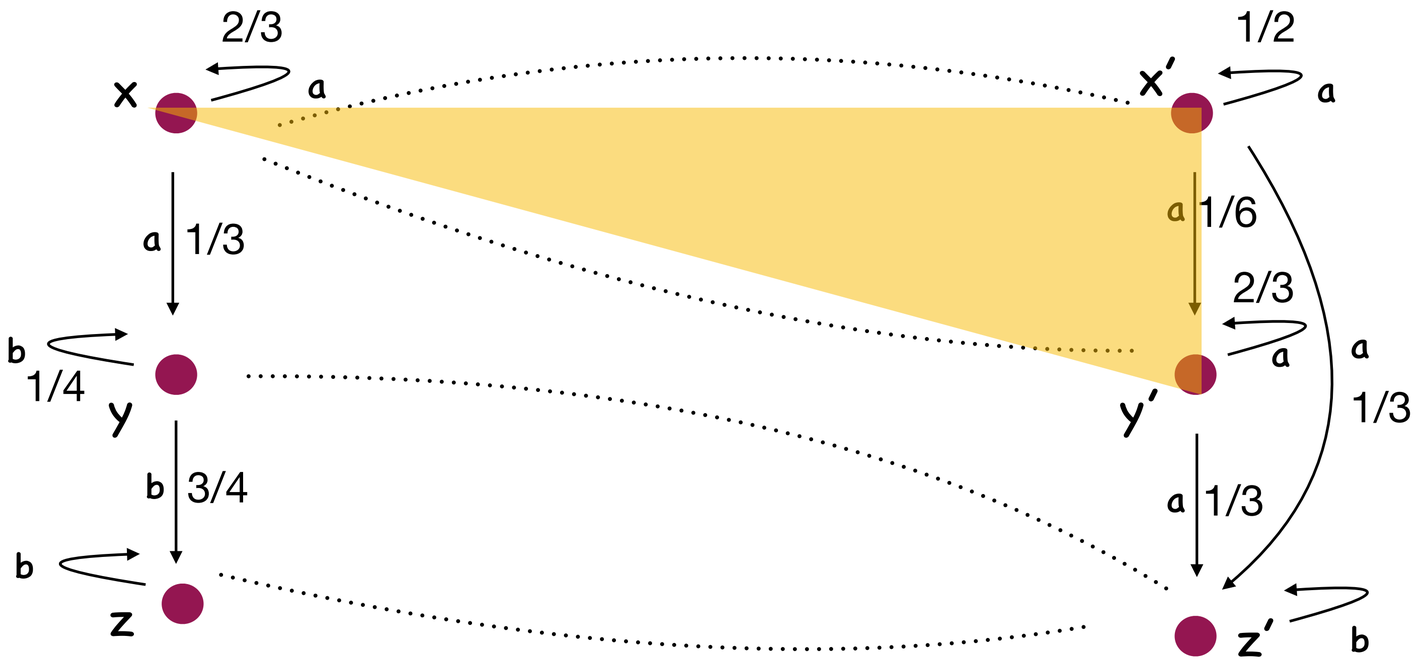

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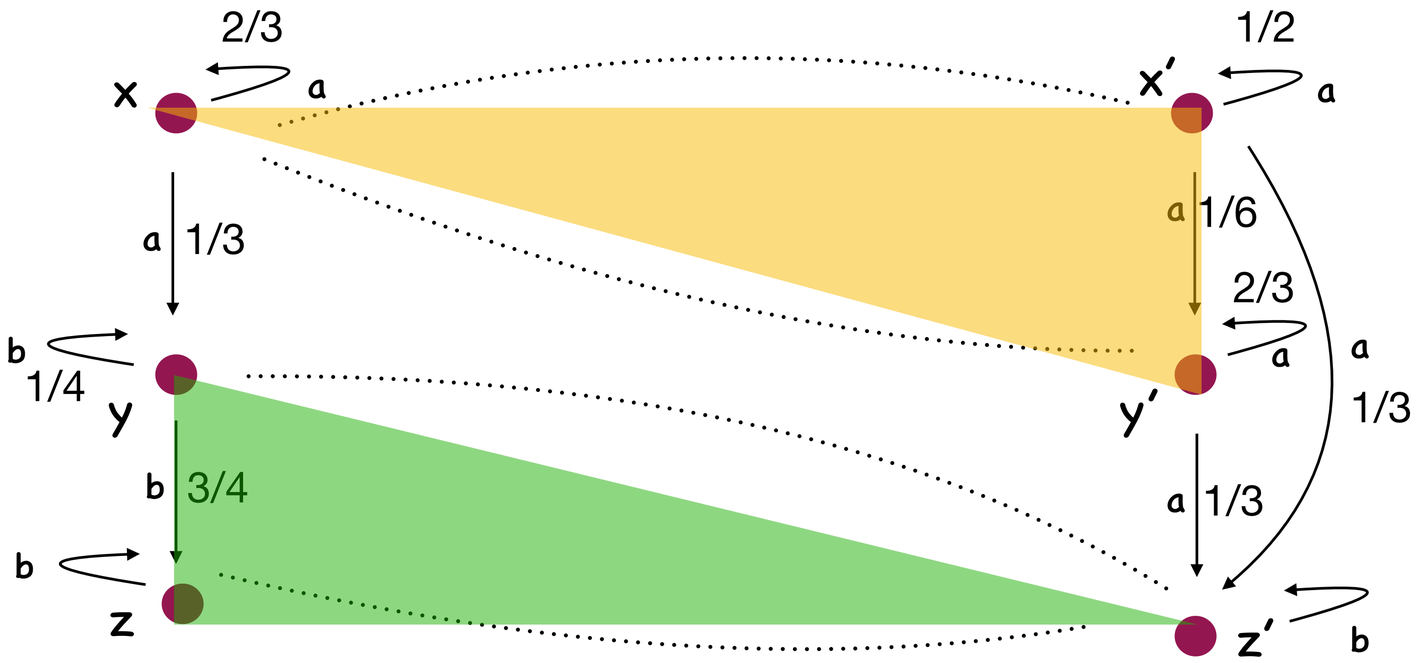
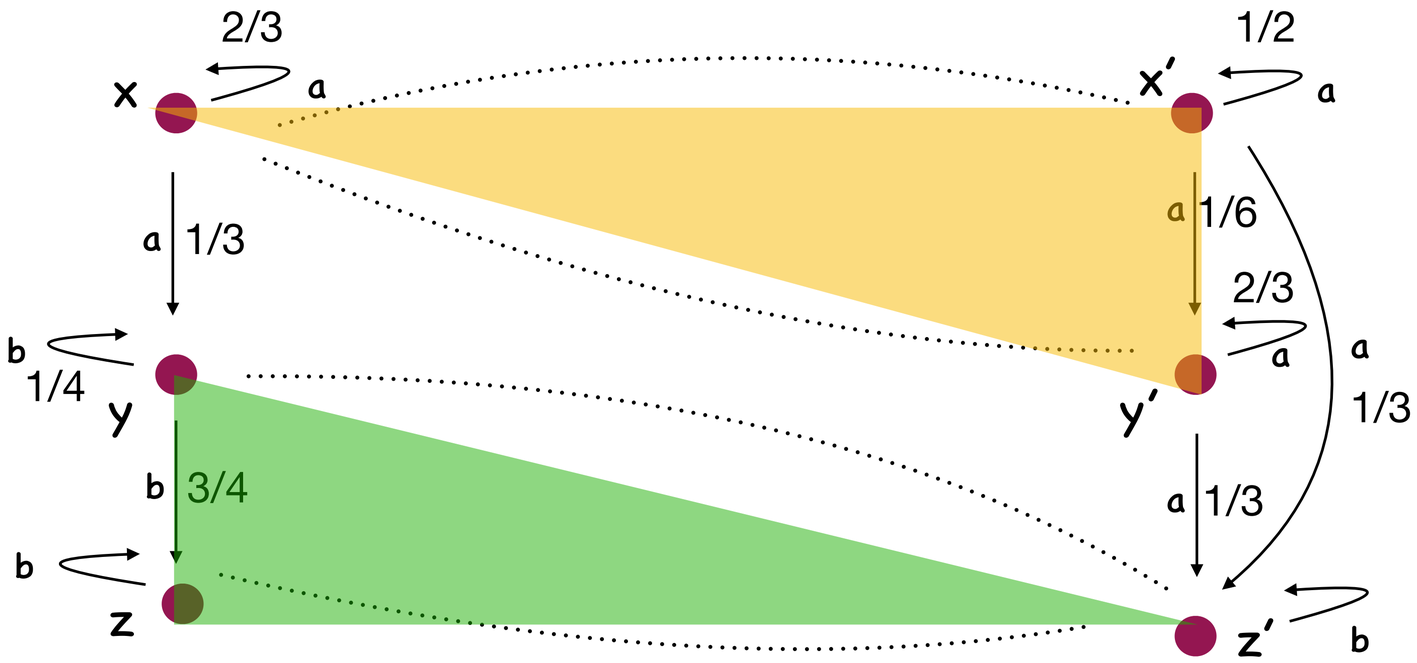


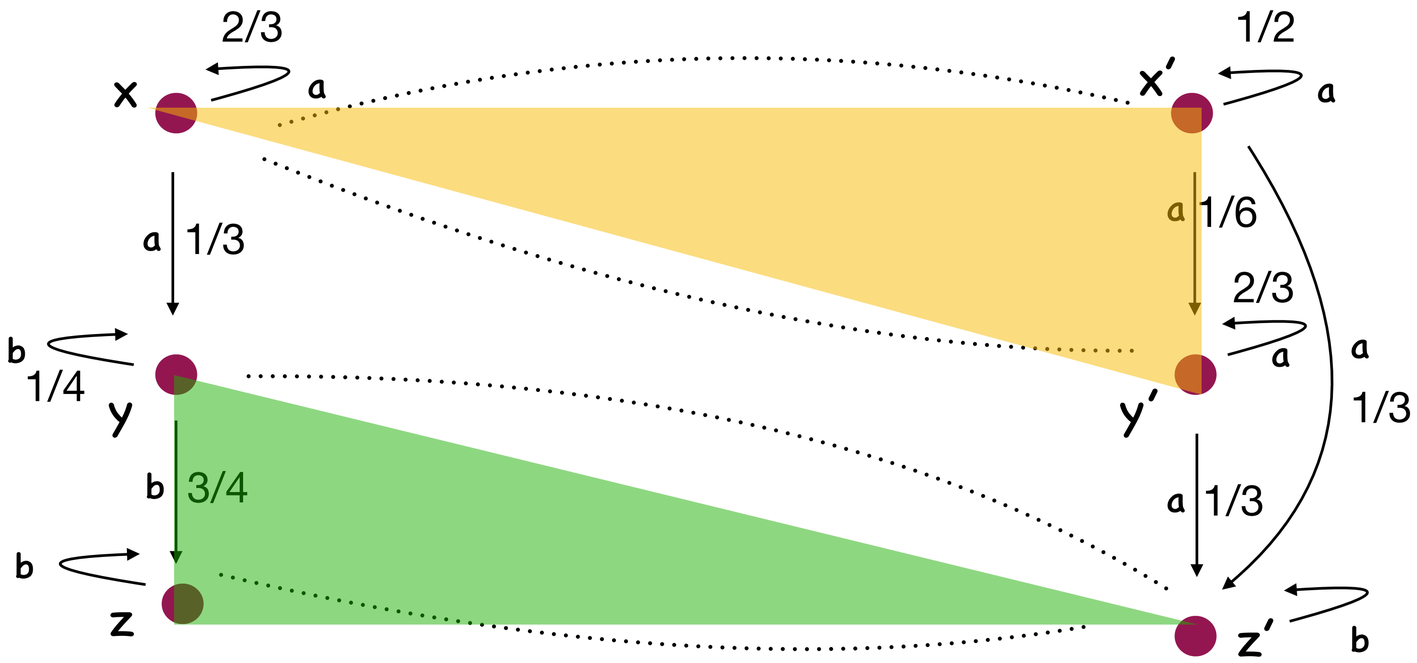
Bisimilarity of Probabilistic Transition Systems



$S \to (\mathcal{D}\, S)^A$

A (probabilistic) bisimulation is an equivalence relation on states preserving probabilities towards equivalence classes.

Bisimilarity of Probabilistic Transition Systems



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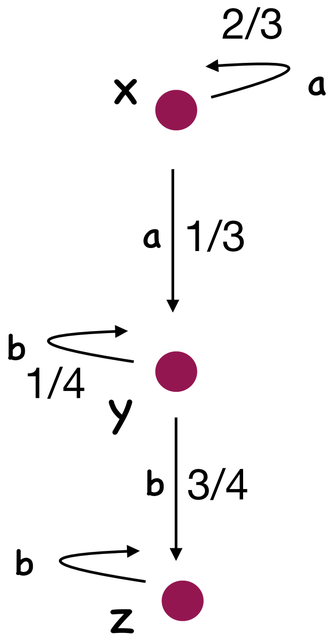
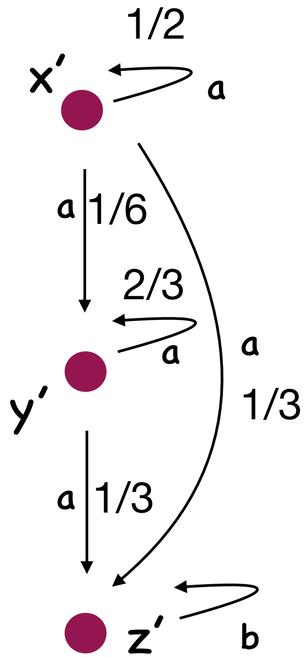
Larsen & Skou 1991

Sometimes bisimilarity is too strong / not robust





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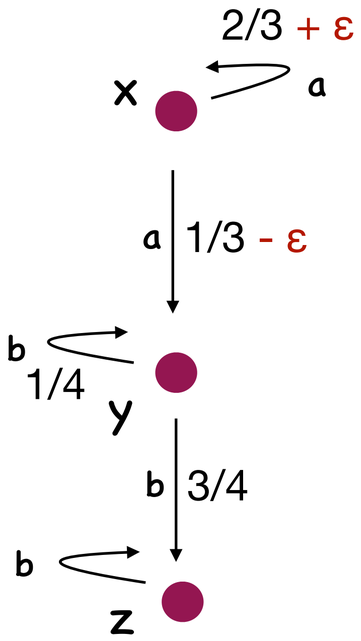

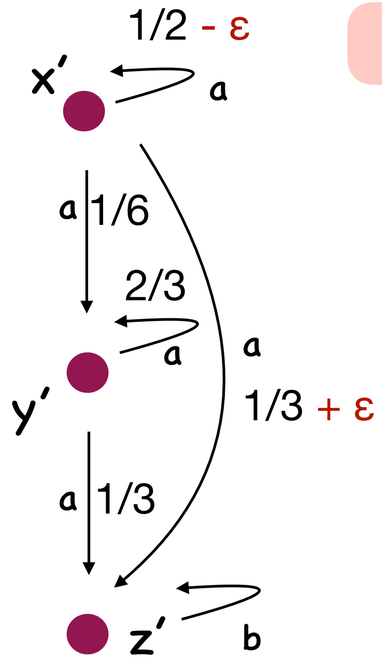

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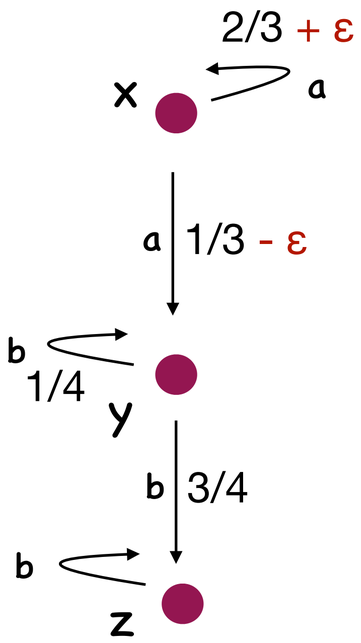

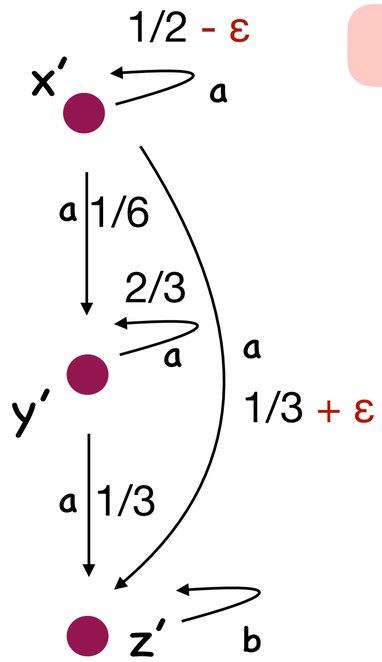

$S \to (\mathcal{D}\, S)^A$

Realistic situation, as probabilities are often approximations of real behaviour.

Two possible solutions:

1. Behavioural pseudo metrics /distances on states, the well known d_K

2. Approximate, ε-bisimulations and ε-bisimilarity

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Desharnais & Panangaden
van Breugel & Worrell
...

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Two possible solutions:

$R_\varepsilon = \{ (x,y) \mid d_K(x,y) < \varepsilon \}$

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$d_\varepsilon(x,y) = \inf\{ \varepsilon \mid (x,y) \text{ is in some } \varepsilon\text{-bisimulation} \}$

...
Desharnais & Laviolette & Tracol
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The well known d_K

Kantorovich distance on distributions

1. Is the gfp of a functional $\Delta(d)(x,y) = \sup_a \delta_K (\tau_a(x), \tau_a(y))$

2. Provides a functor, even monad on .. pseudo metric spaces, has a final coalgebra.

3. Has a generic categorical lifting.

4. The monad is axiomatizable by a quantitative theory.

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Has no intuitive approximate bisimulation.

ε-Bisimulation, ε-distance (one label)

...
Desharnais & Laviolette & Tracol
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ε-Bisimulation, ε-distance (one label)

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Desharnais & Laviolette & Tracol
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1. A relation R is an ε-simulation iff $x\ R\ y \Rightarrow \tau(x)(X) \leq \tau(y)(R(X)) + \varepsilon$ for any subset of states X.

2. It is an ε-bisimulation if it is a symmetric ε-simulation.

3. $s \sim_\varepsilon t$ if they are related by some ε-bisimulation.

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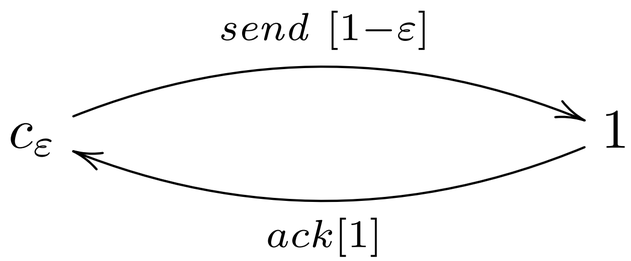
Has no fixpoint characterisation, or other general properties ?

ε-simulation, ε-bisimulation example



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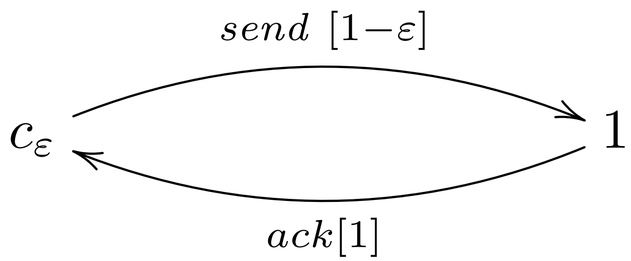
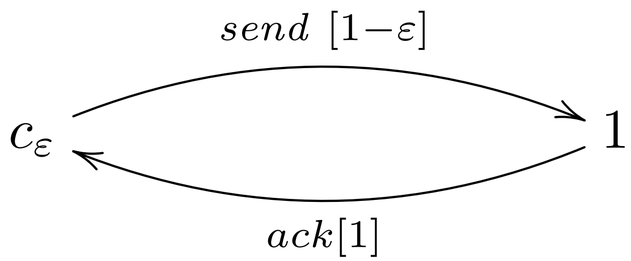


The ε-distance between correct and approximate channel is indeed ε

ε-simulation, ε-bisimulation example

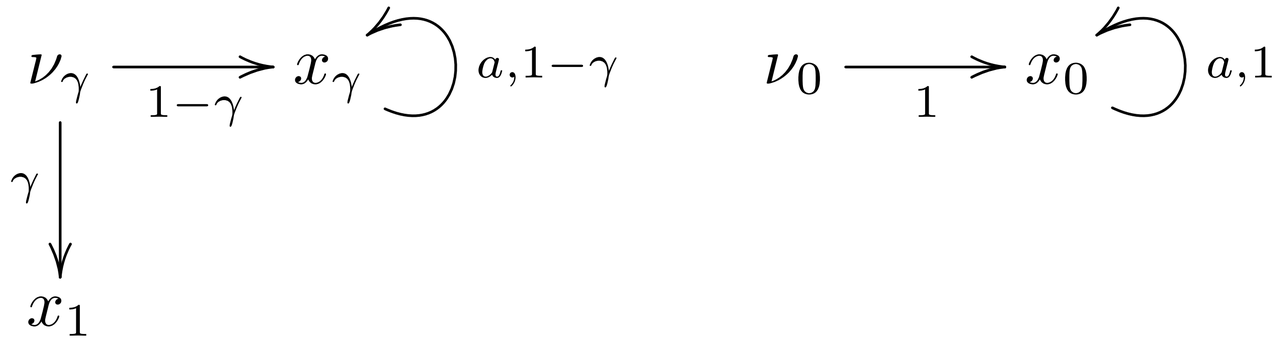


The ε-distance between correct and approximate channel is indeed ε

The K-distance between correct and approximate channel is 1

ε-simulation, ε-bisimulation example

$\gamma \in [0,1/2)$



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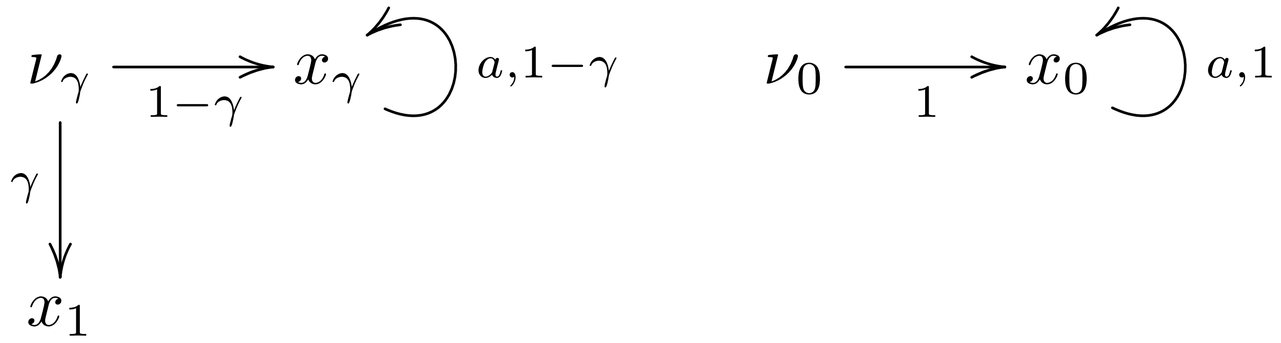

The ε-distance between ν_γ and ν_0 is γ

ε-simulation, ε-bisimulation example

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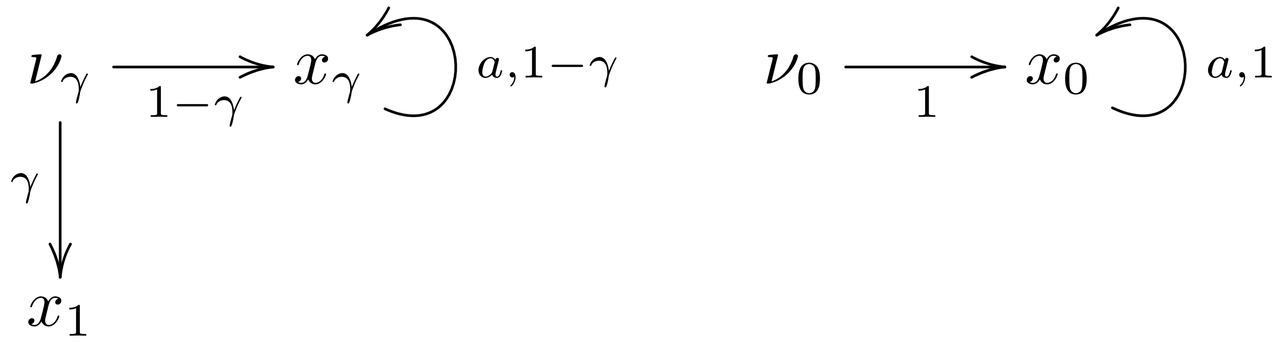

The ε-distance between ν_γ and ν_0 is γ

The ε-distance between ν_γ and ν_ξ is $|\gamma-\xi|$

Our new results on ε-distance:

Josée Desharnais

Desharnais & S. 2025

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1. It is the gfp of a functional $\Delta(d)(x,y) = \sup_a \delta_{LP,d}(\tau_a(x), \tau_a(y))$
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Lévy-Prokhorov lifting

Starting from a metric space (S,d), it gives a distance on $\mathcal{D}$S by

$$\delta^d_{LP}(\mu_0, \mu_1) = \inf\{\varepsilon \mid \forall X \subseteq S : \mu_i(X) \leq \mu_{1-i}(X^d_\varepsilon) + \varepsilon,\ \ for\ i = 0, 1\}$$

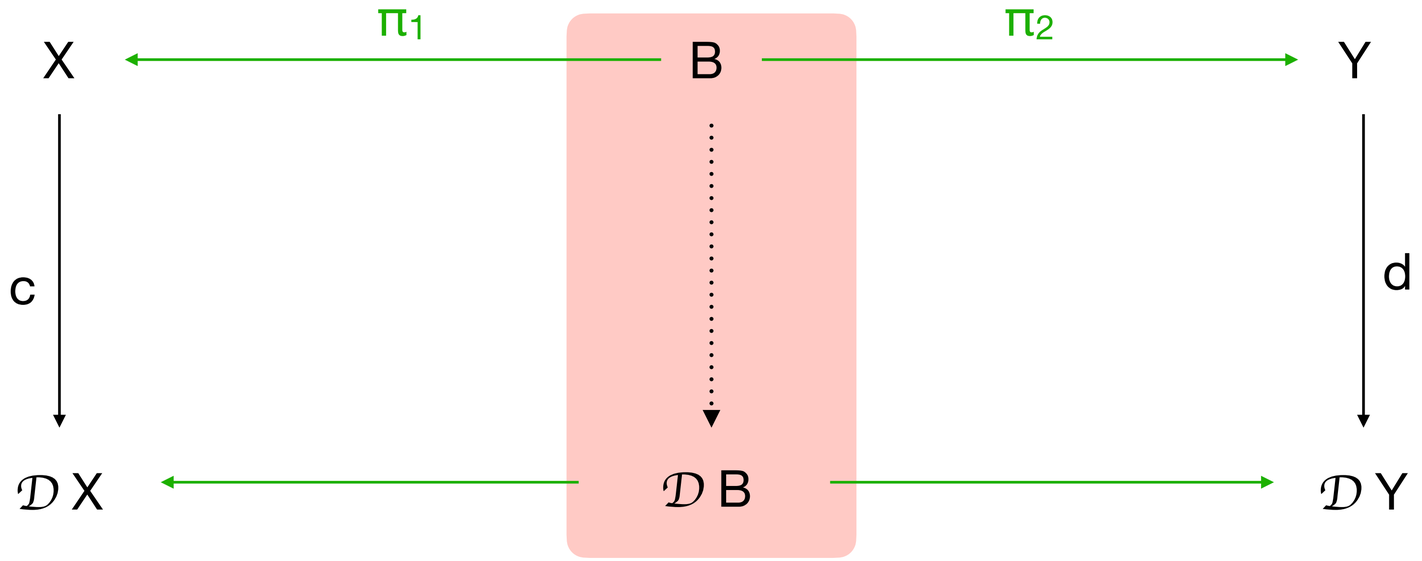
where

$$X^d_\varepsilon = \{y \mid \exists x \in X : d(x, y) < \varepsilon\}$$

... and further a behavioural distance on $S \to (\mathcal{D}\, S)^A$ **by fixpoints**

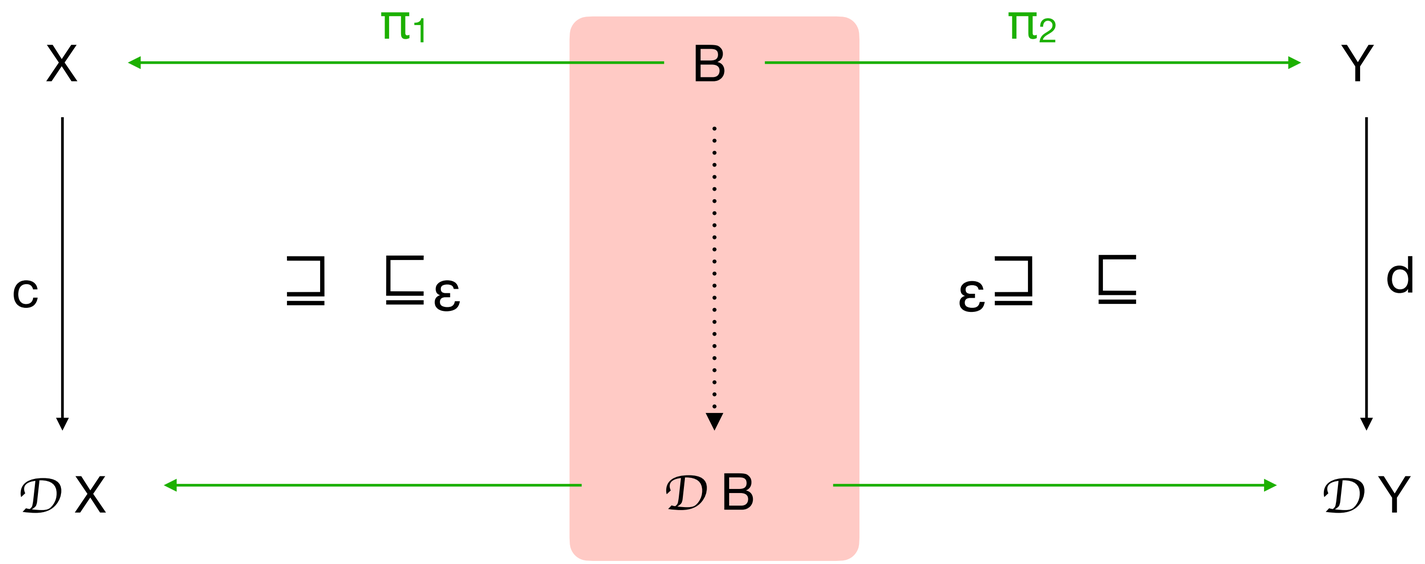
Recall bisimulation for PTS (one label)

$S \to \mathcal{D}\, S$



ε-Bisimulation in a diagram

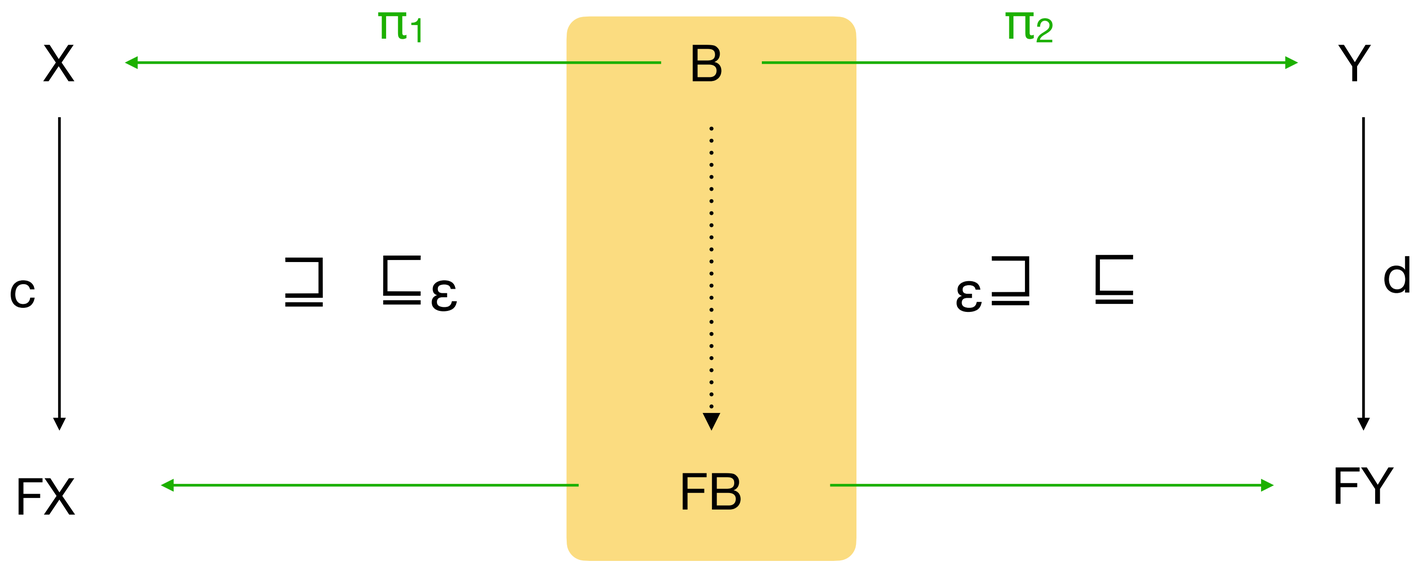
$S \rightarrow \mathcal{D}\,S$



with: $\varphi \sqsubseteq_\varepsilon \psi$ iff for all $A \subseteq B$, $\varphi(A) \leq \psi(A) + \varepsilon$

ε-Bisimulation for F-coalgebras

$S \to FS$

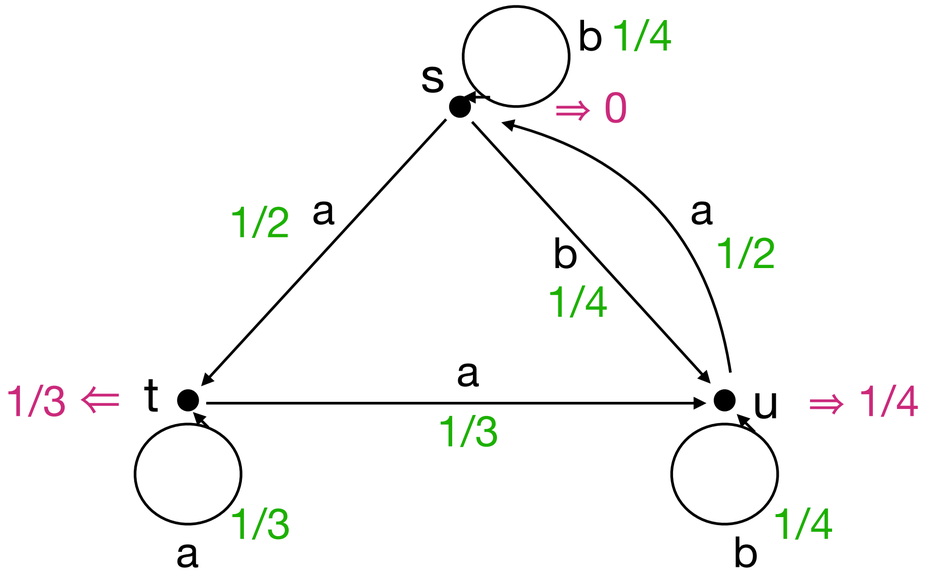


as long as F comes with a suitable order

this enables new definition of approximate bisimulations for other types of systems

Trace Semantics = Bisimilarity Elsewhere

Finite Traces



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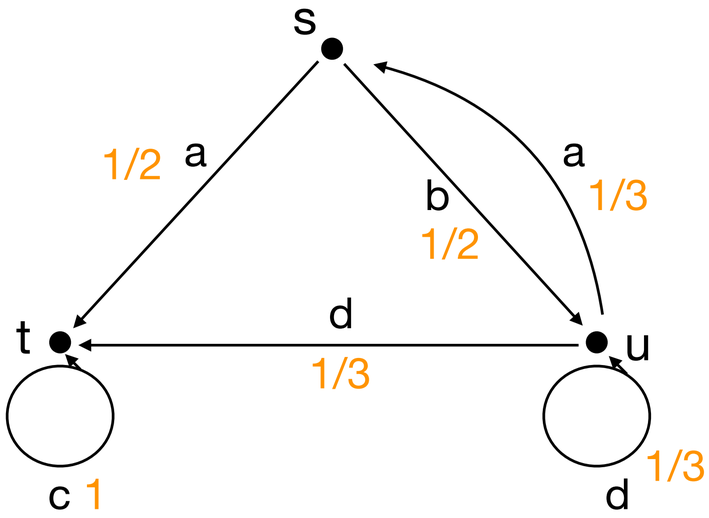
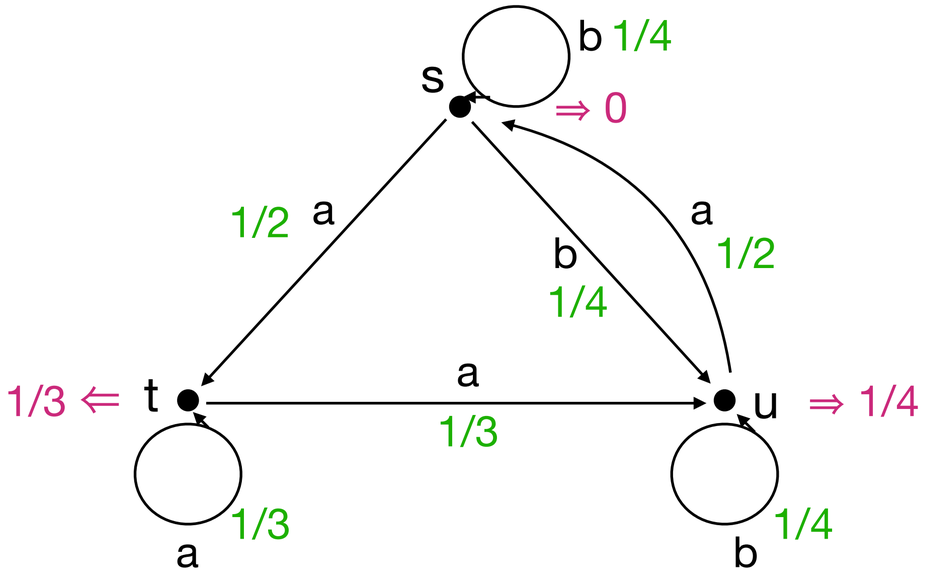


Finite Traces

c: S→ $\mathcal{D}_{\leq}$ (A x S)



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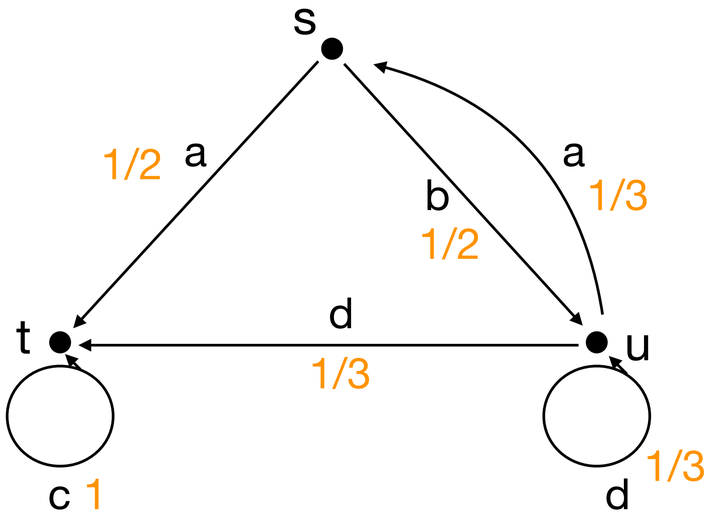

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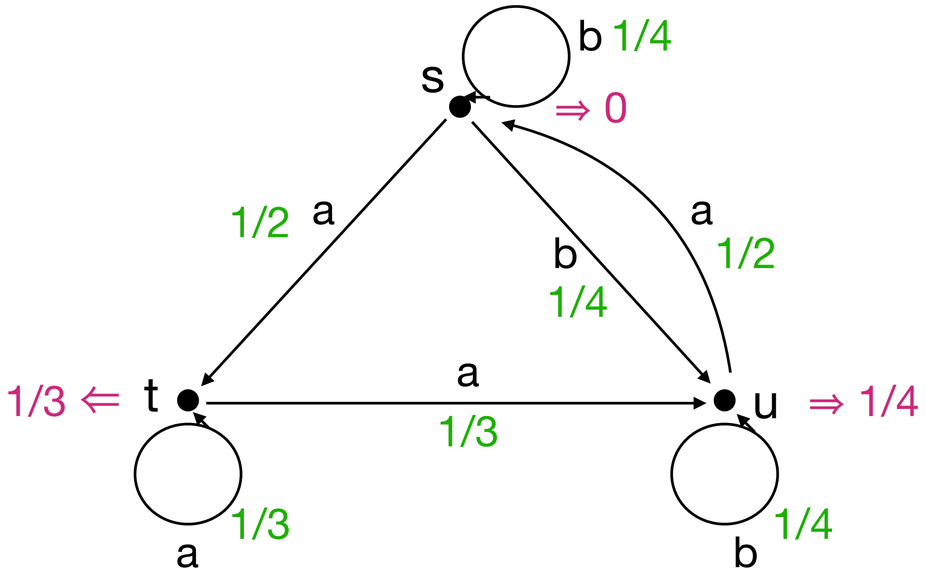

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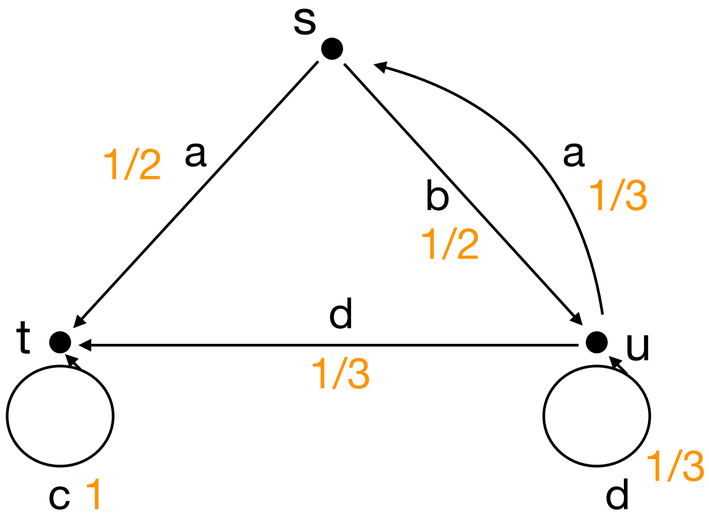

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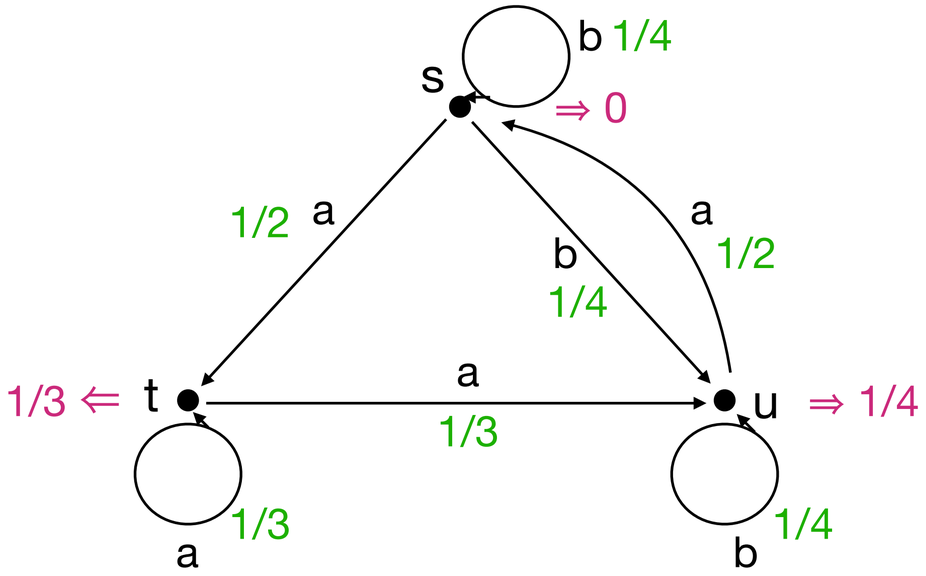

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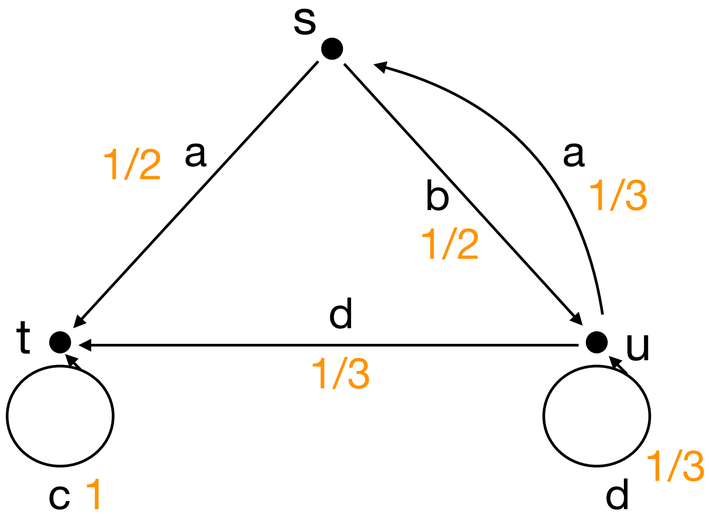

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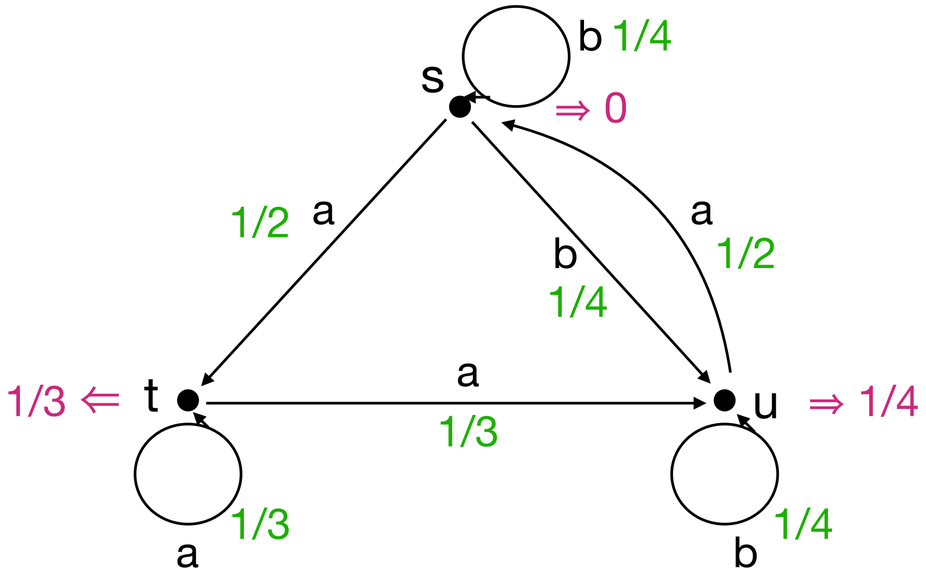

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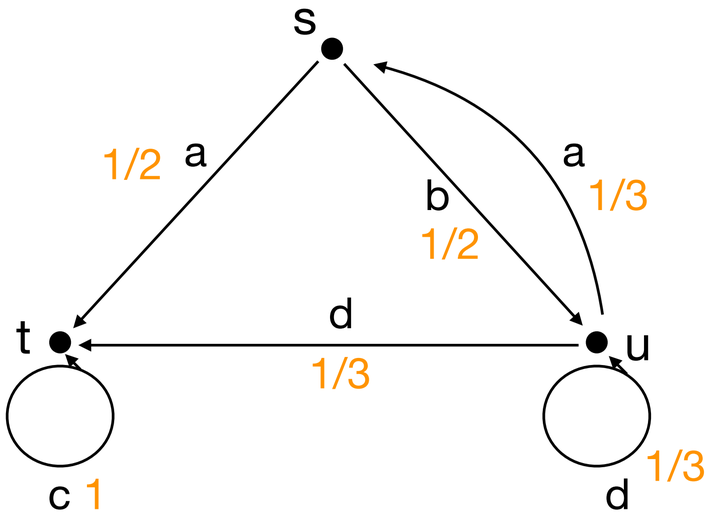

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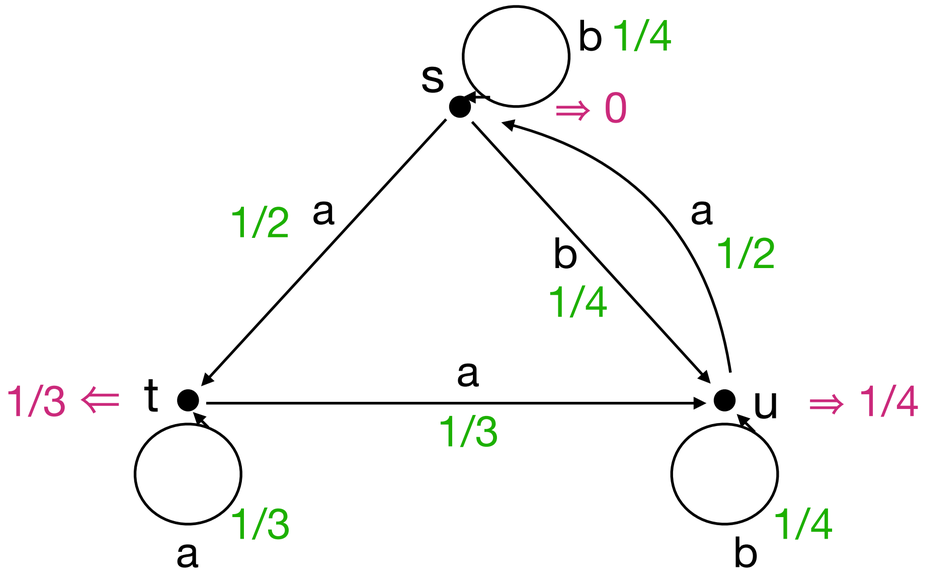

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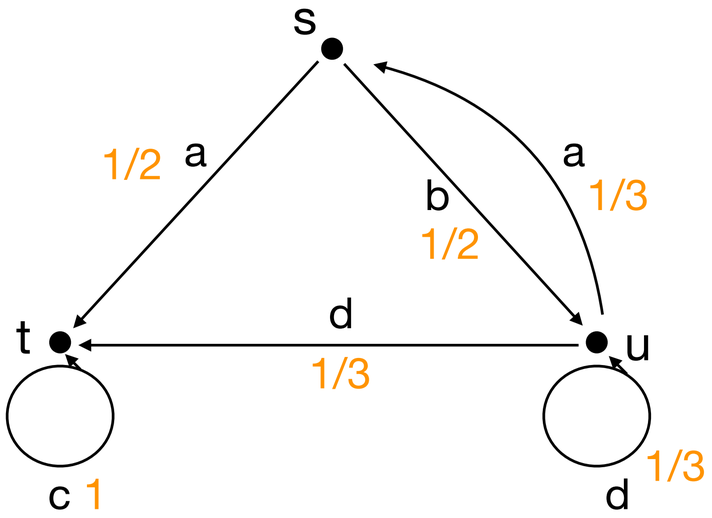

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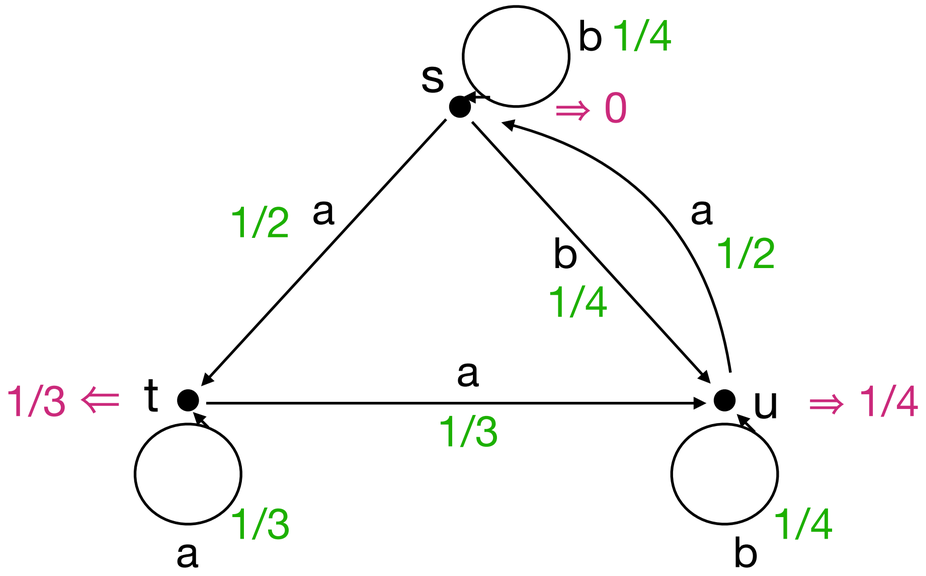

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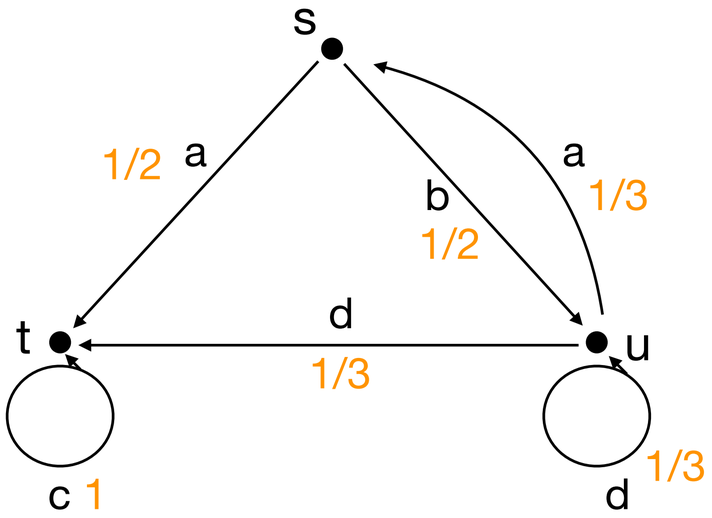

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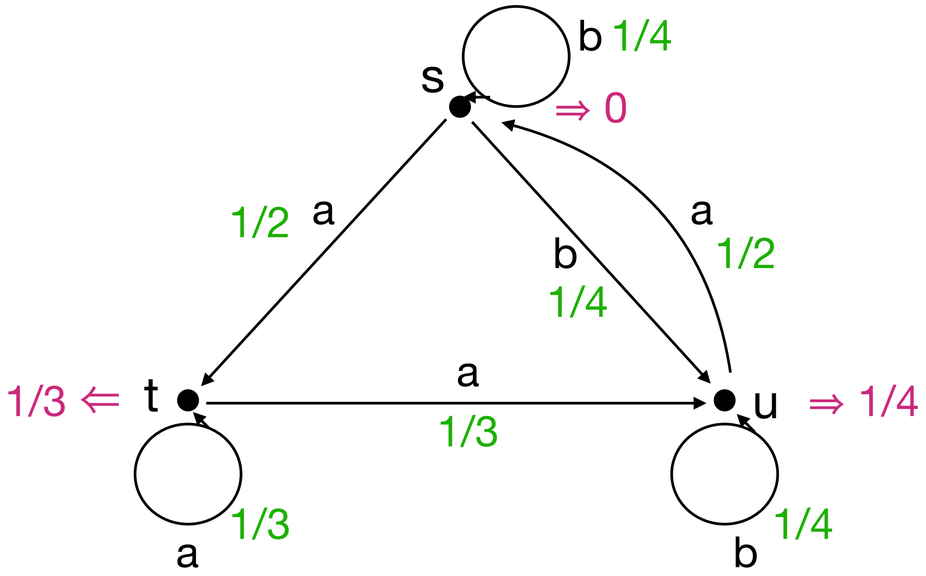

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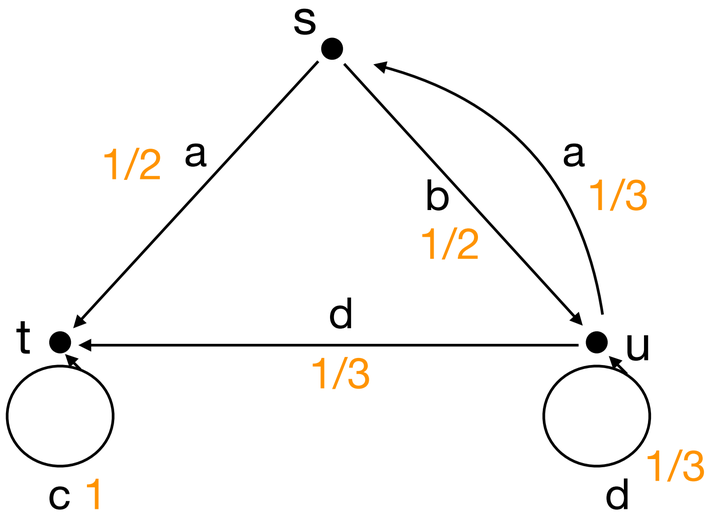

trace (s) (ac^{ω}) = 1/2

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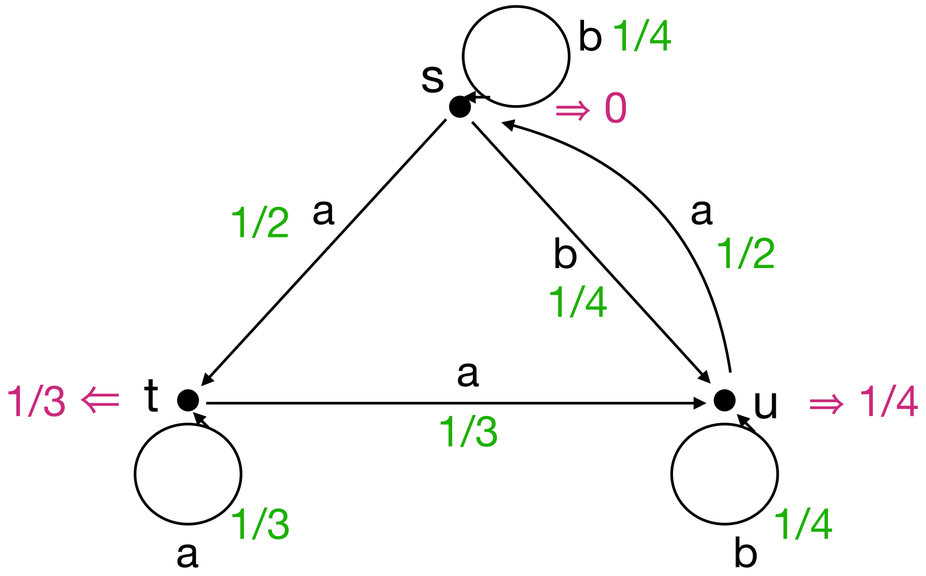

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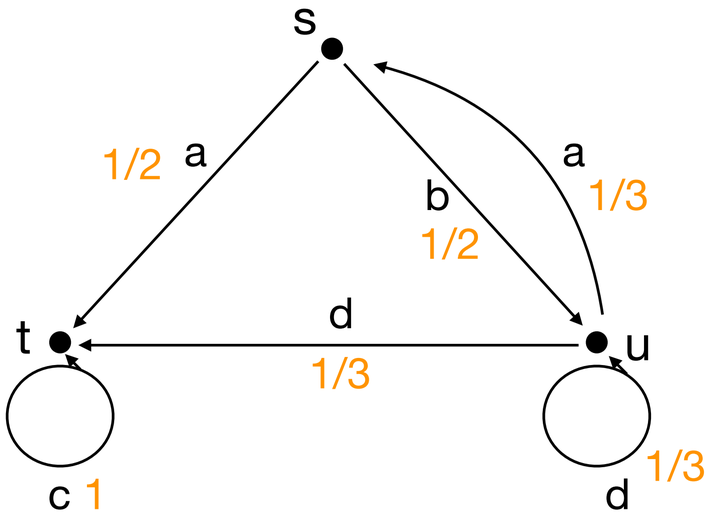

trace (s) (ac^{ω}) = 1/2

trace (s) (bd^{ω}) = $\lim_{n\to\infty}$ 1/2 · $(1/3)^n$ = 0

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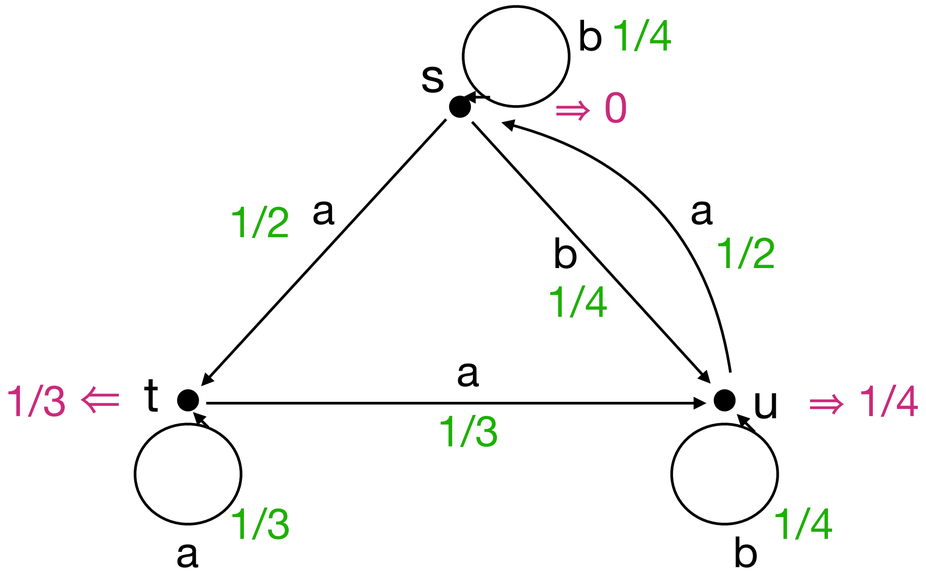

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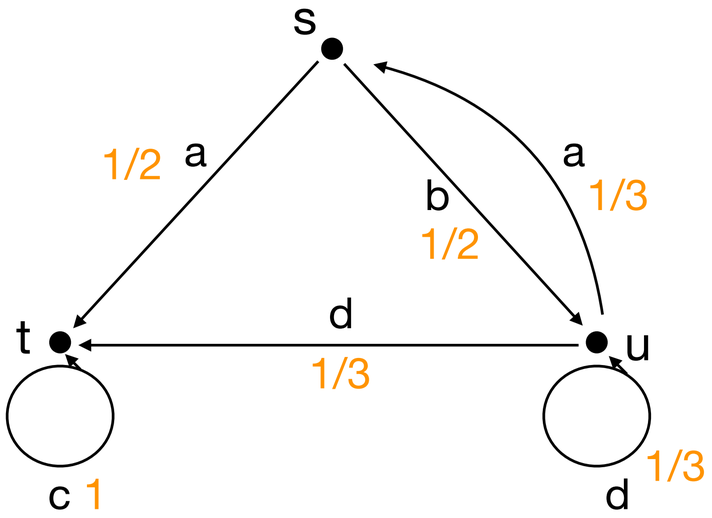

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Therefore, measurable spaces !

Finite Traces

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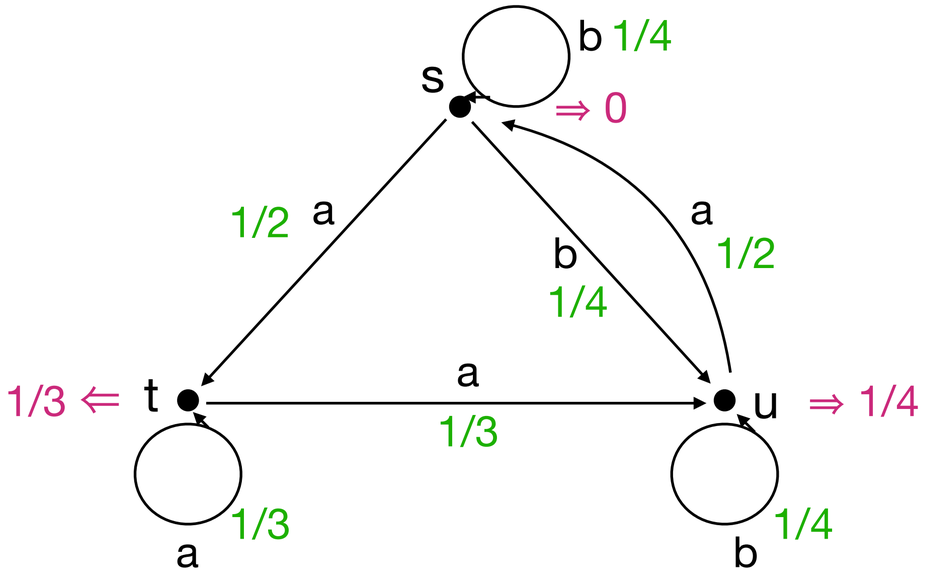

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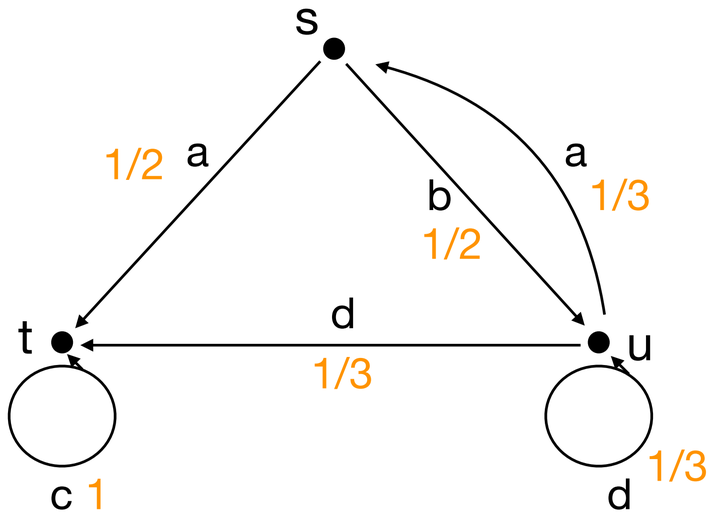

Each state defines a Borel probability measure on A^{ω}, trace (s) ∈ Prob(A^{ω})

with the σ-algebra generated by the cylinders

$B_w = \{ \alpha \in A^{\omega} \mid w \text{ is a prefix of } \alpha \}$

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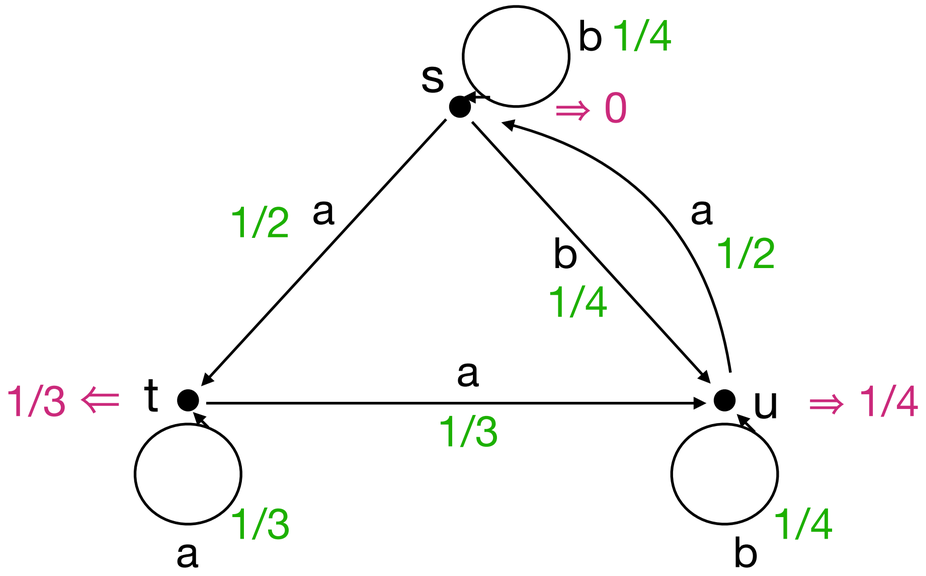

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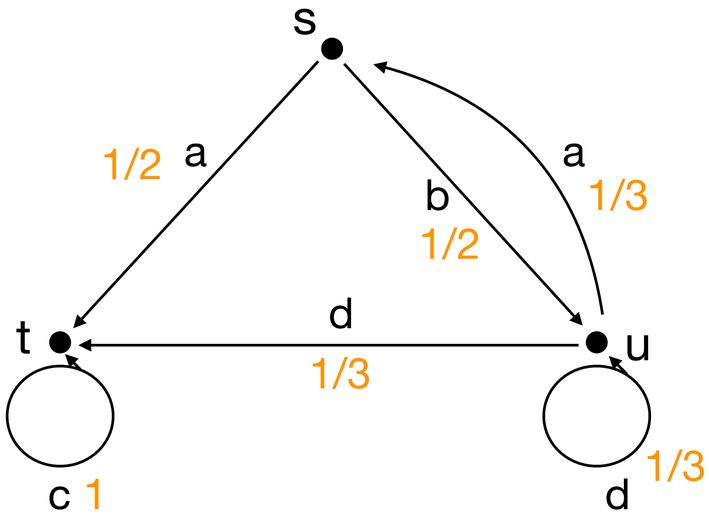

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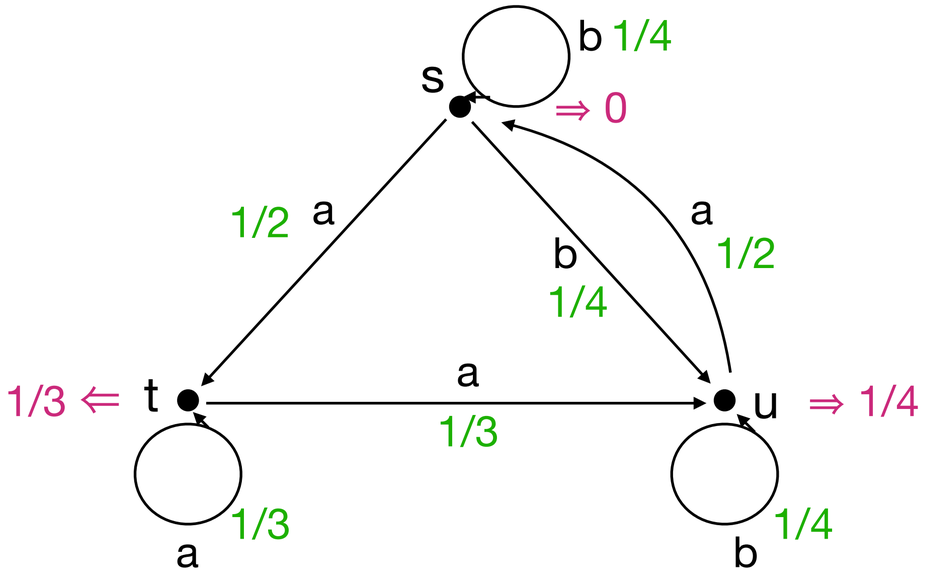

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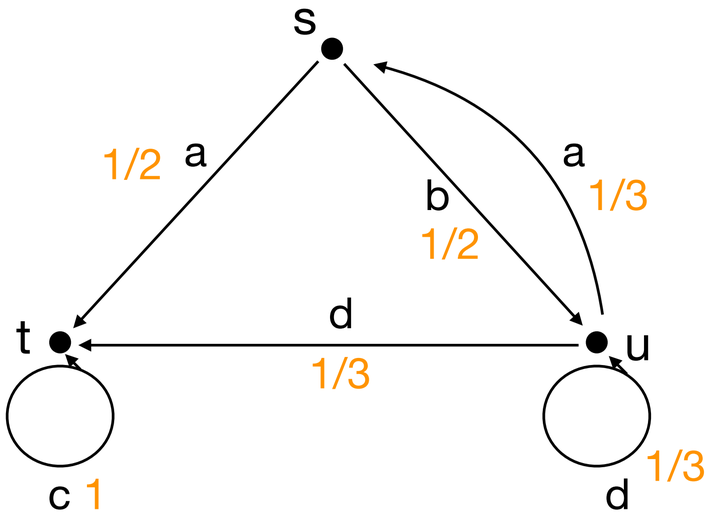

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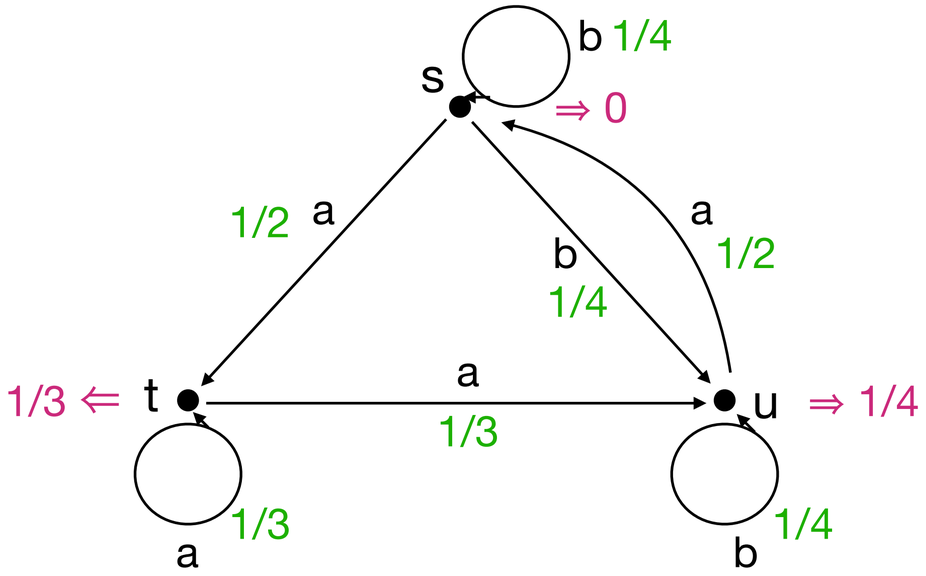

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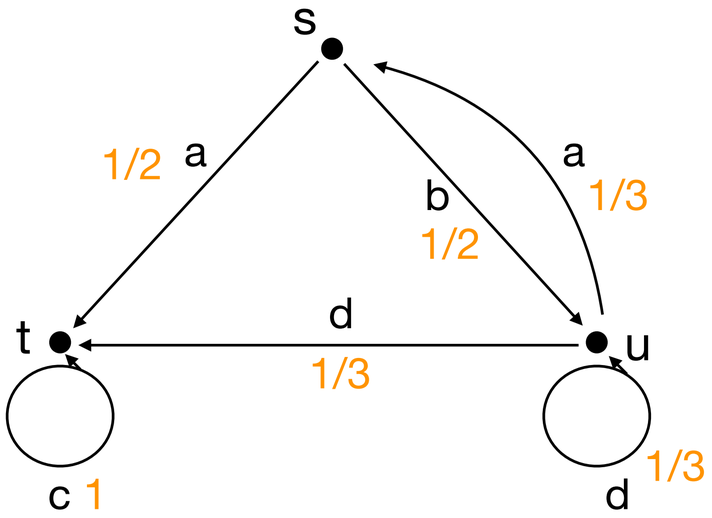

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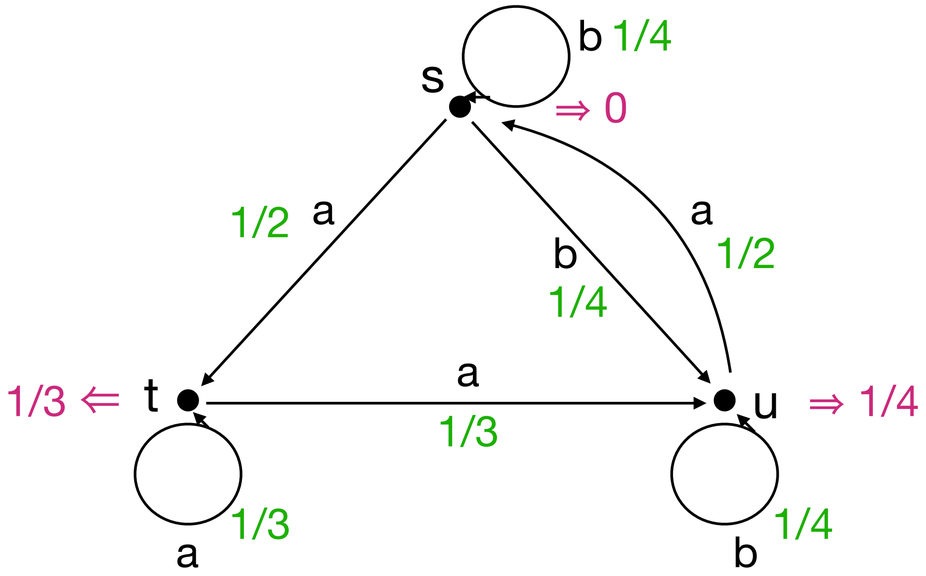

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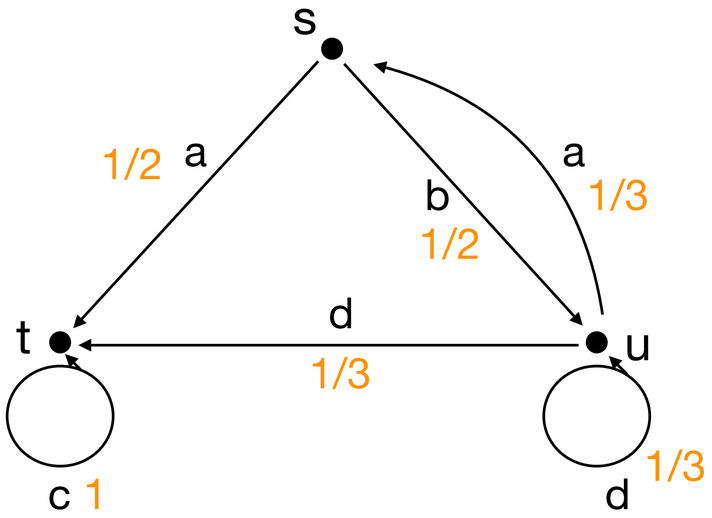

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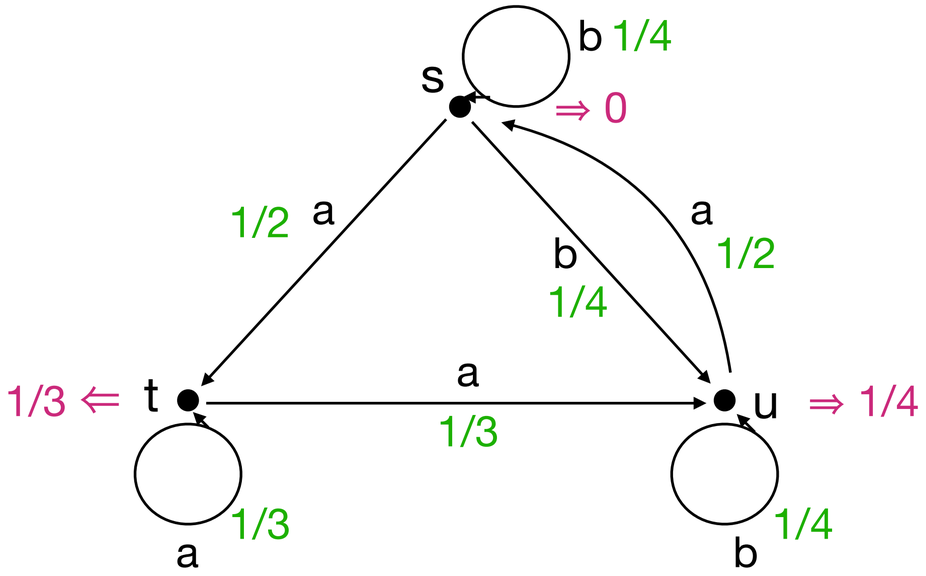

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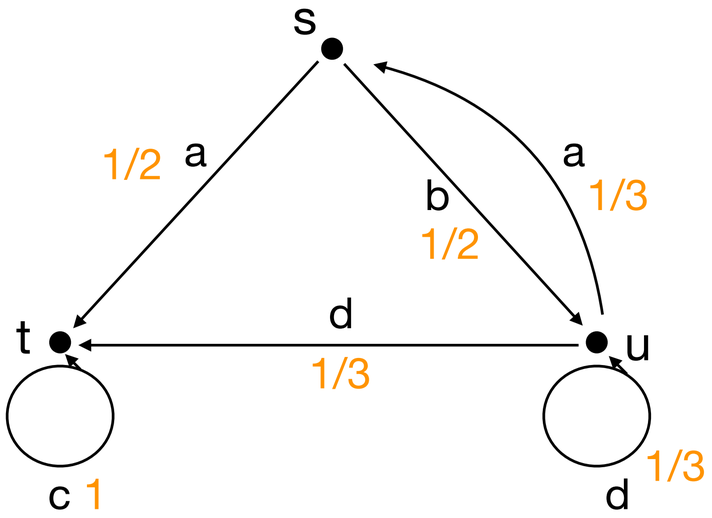

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Two states s and t are trace equivalent, notation $s =_{tr} t$, if and only if they have the same trace:

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Theorem (Kleene-style) : For every state s in an LMC, there exists a process term t_s with

$$s =_{tr} t_s$$

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Theorem (Soundness&Completeness):

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Theorem (Soundness&Completeness):

Two states s and t in an LMC are trace equivalent iff their corresponding process terms are provably equivalent

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convex algebras

Basically the same from Silva & S. ‘11

$a.\ (e \oplus_p f) = a.e \oplus_p a.f$

distributivity

$\mathrm{fix}\ v.e = e[v := \mathrm{fix}\ v.e\ ]$

$g = e[v := g] \Rightarrow\ g = \mathrm{fix}\ v.e$

fixpoints

Theorem (Soundness&Completeness):

Two states s and t in an LMC are trace equivalent iff their corresponding process terms are provably equivalent

$$s =_{tr} t \iff \vdash t_s = t_t$$

The Role of Algebras

A lot of work happens in categories of algebras, for LMC - convex algebras

Eilenberg-Moore algebras for $\mathcal{D}$

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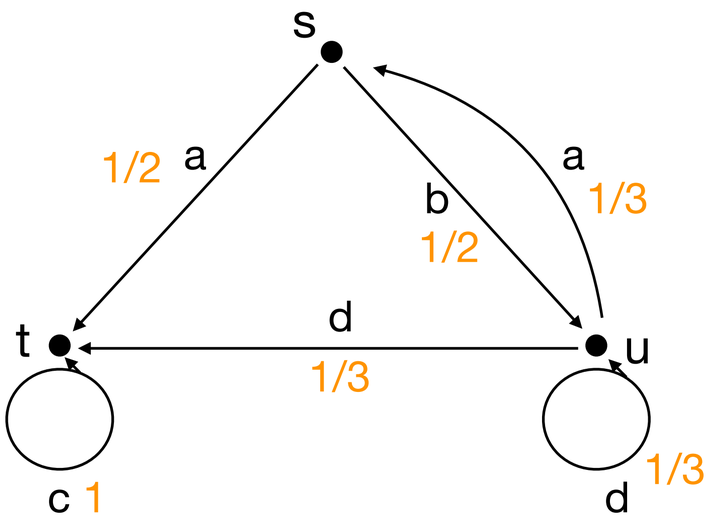

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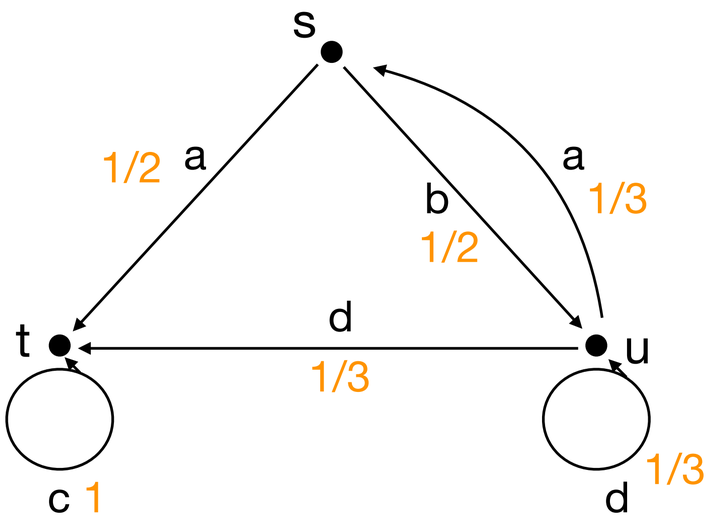

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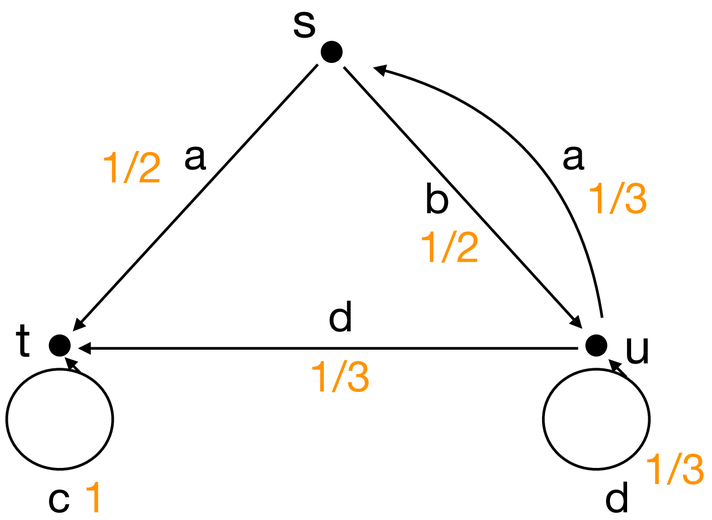

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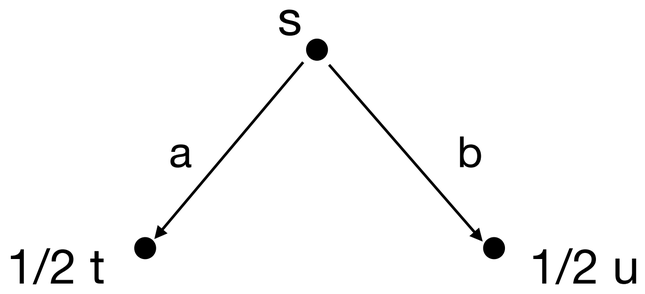

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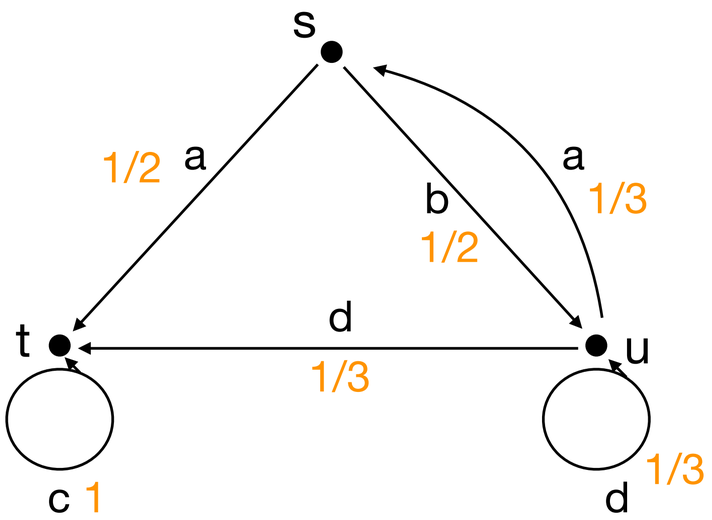

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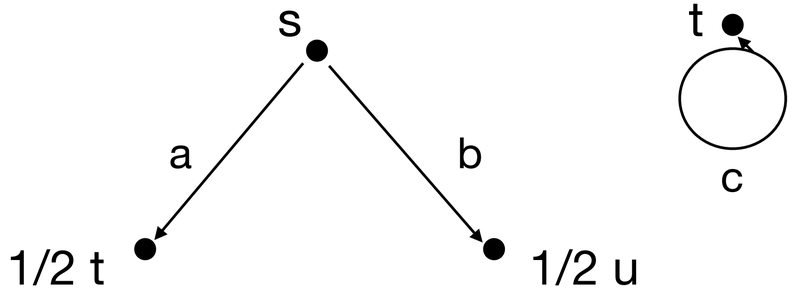

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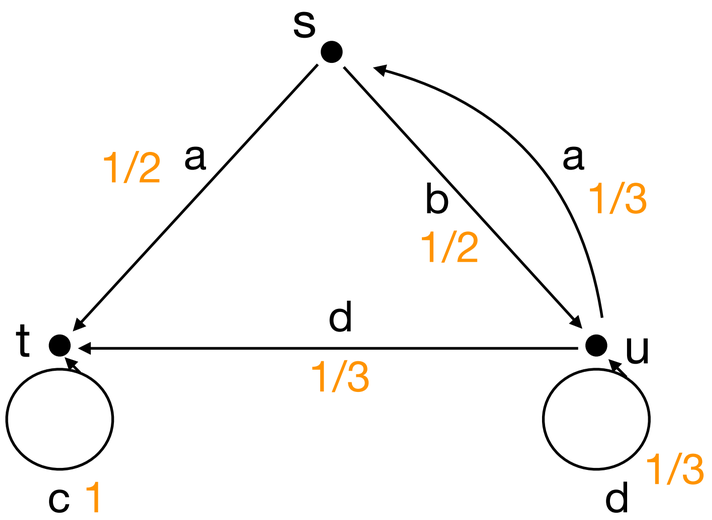

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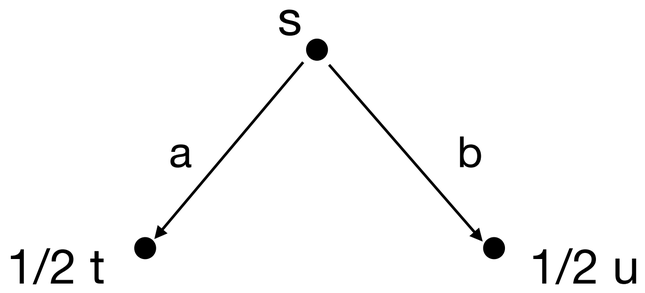

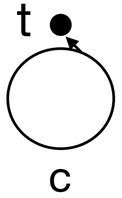

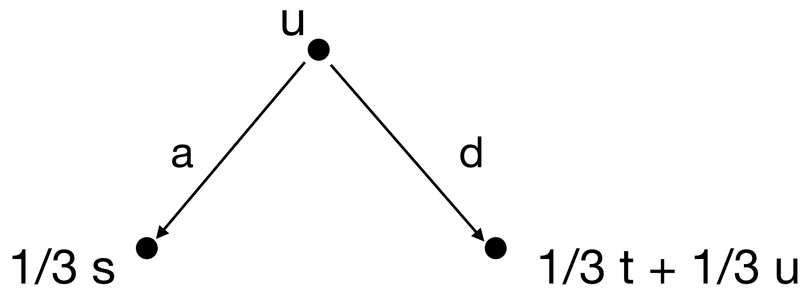

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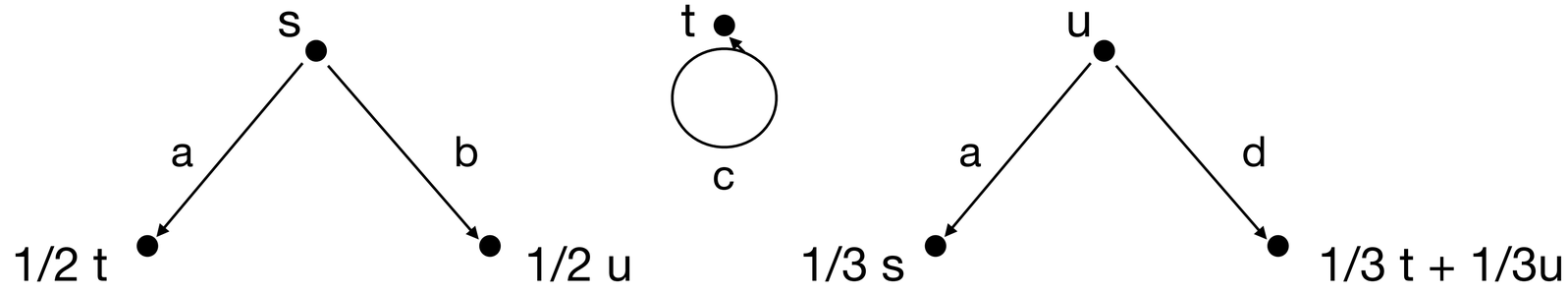

There are different ways to represent this as a coalgebra $\mathcal{D}_{\leq}\,S \to G\mathcal{D}_{\leq}\,S$

The role of G

1. Every coalgebra c: S → $\mathcal{D}$ (A x S)
   determinizes to c#: $\mathcal{D}$S → G$\mathcal{D}$S

2. Prob(A^ω) carries the final
   G-coalgebra

   Prob(A^ω) → G Prob(A^ω)

3. Infinite trace equivalence
   is the final coalgebra semantics

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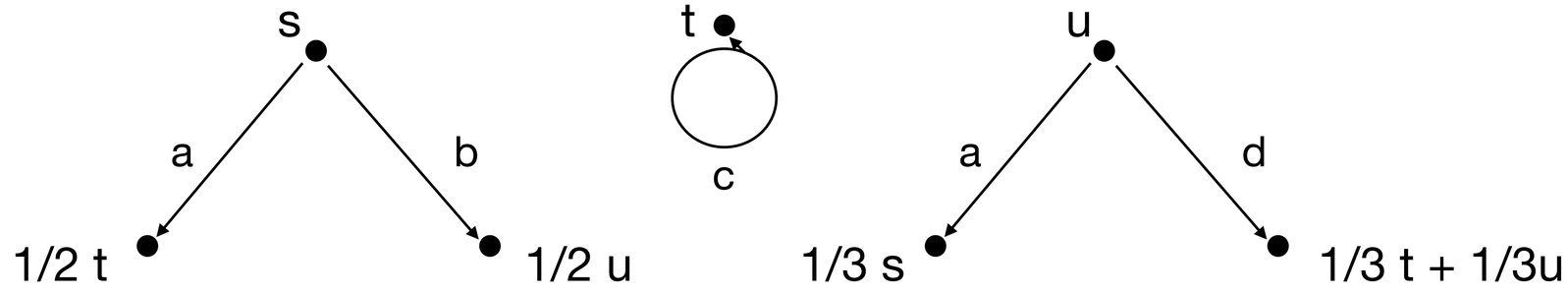

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First functor on Convex Algebras:

$$X_{\perp} = \{\perp\} \cup \{r \cdot x \mid r \in (0,1], x \in X\}$$

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Idea (total weight) from Goy & Rot ‘18

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Proof obligations:

G is finitary, preserves surjective homomorphisms,… is proper !

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It relates x ∈X and y ∈Y, denoted x ~ y, if there exist $z_{2k} \in Z_{2k}$, for k = 1,.., n-1 with $z_0 = x$, $z_{2n} = y$,

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F is proper if whenever X and Y have free fg carriers, and x ~ y, then there is a zigzag whose all nodes have free fg carriers.

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F is proper if whenever X and Y have free fg carriers, and $x \sim y$, then there is a zigzag whose all nodes have free fg carriers.

For the upper nodes, it suffices to be fg.

Some More History

Milius ’12 If fp = fg and  , then “completeness”

S. & Woracek ’15 fp = fg for convex algebras, but no 

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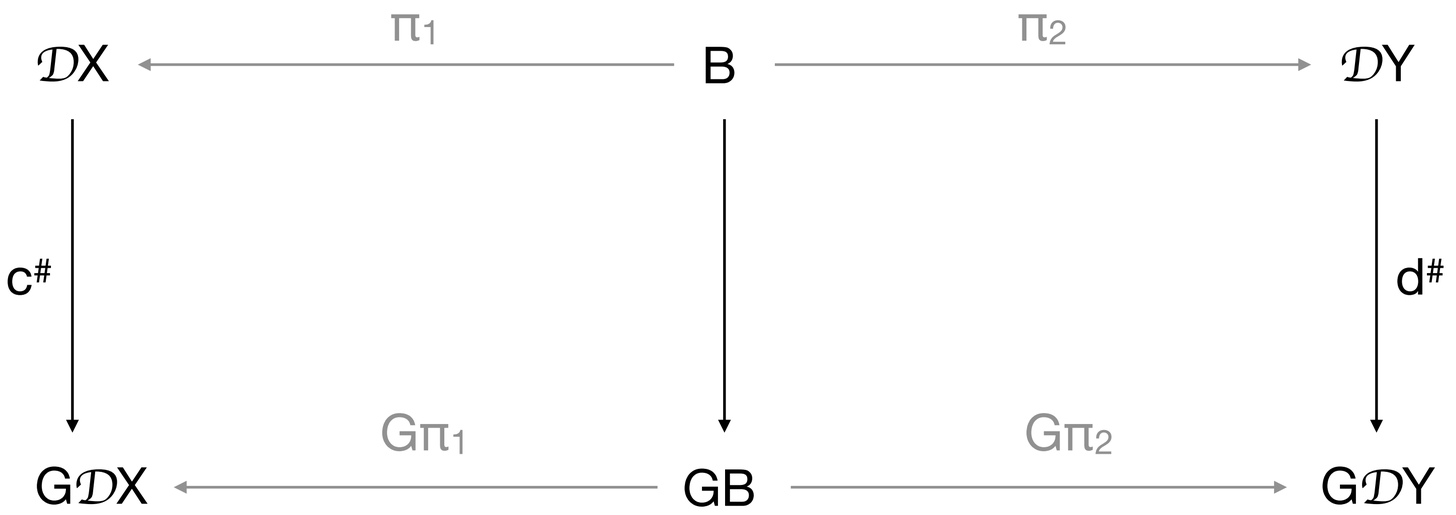
Bisimilarity gives the zigzag

Bisimilarity Gives the Zigzag



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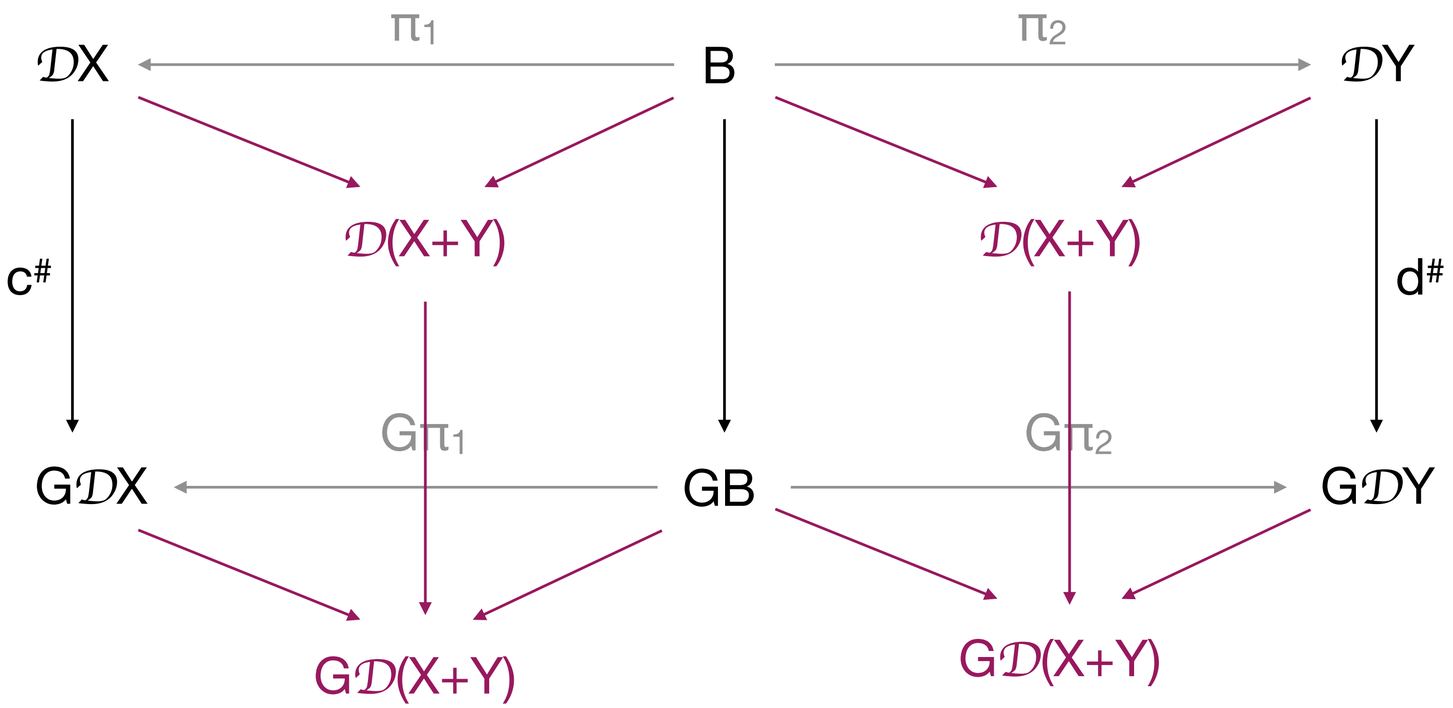

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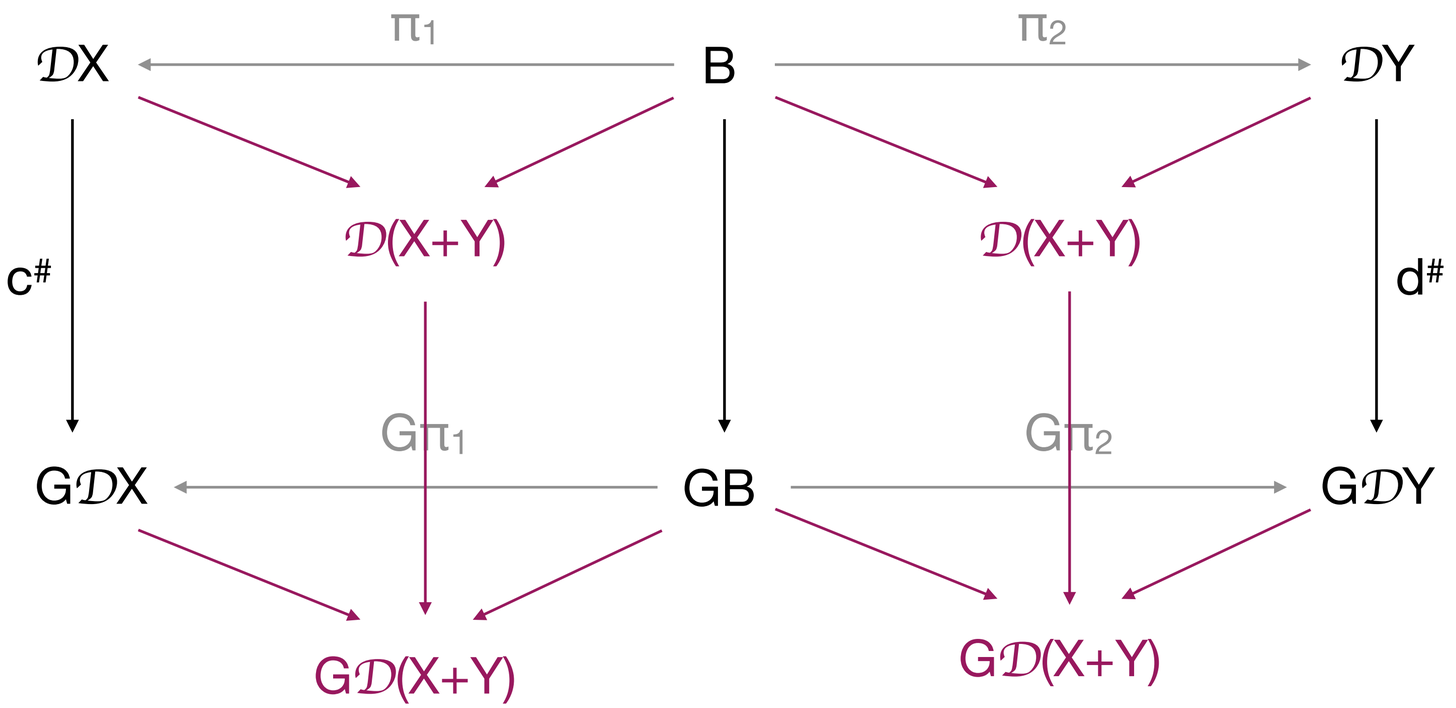

S. & Woracek ’15 B is fg as a subalgebra iff it is topologically closed in $\mathbb{R}^{2|X+Y|}$

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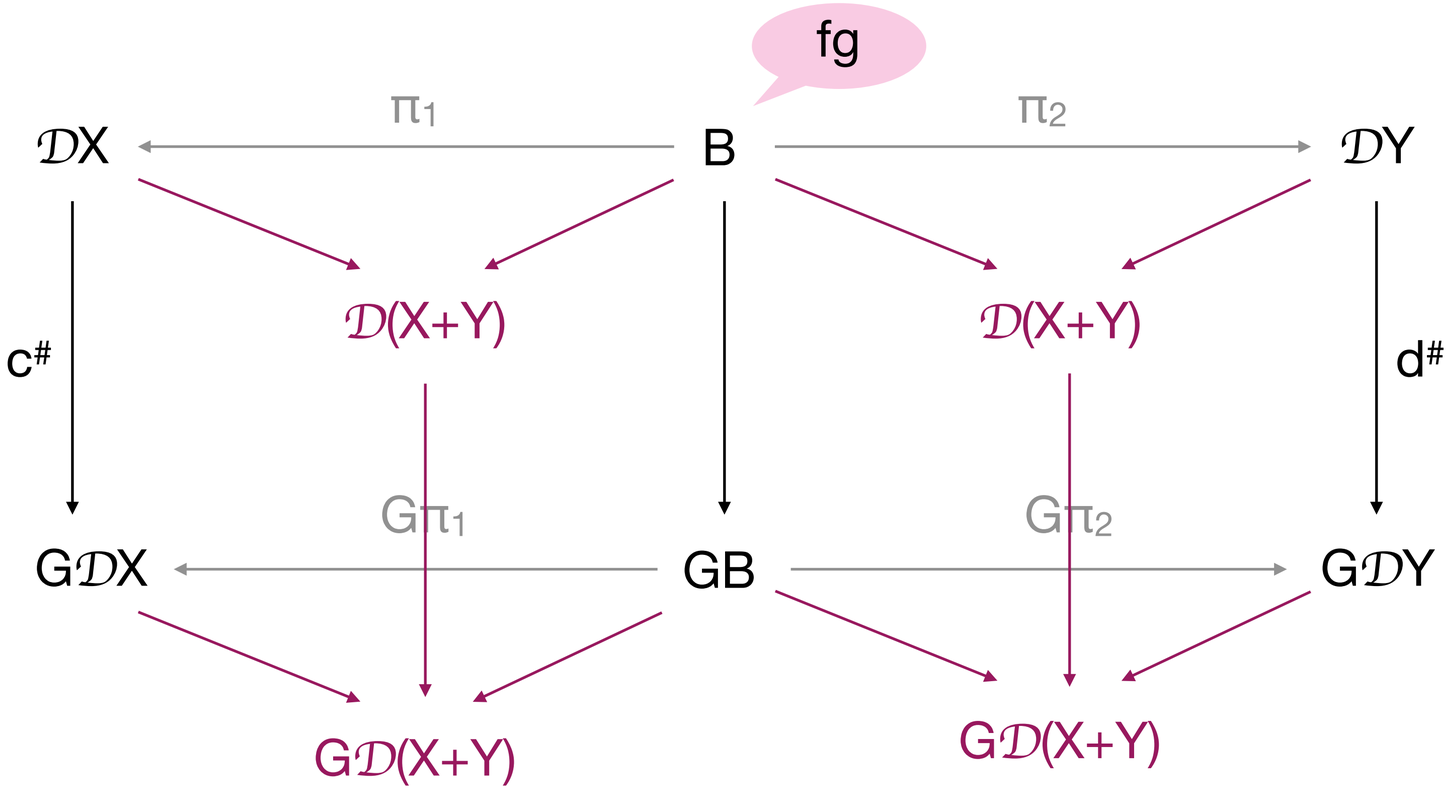

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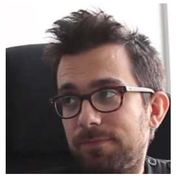
Thank You and Thanks to:



Filippo Bonchi

Corina Cirstea

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Ichiro Hasuo

Bart Jacobs

Larry Moss

Tori Noquez

Todd Schmidt

Alexandra Silva

Valeria Vignudelli

Erik de Vink **TU/e**

Harald Woracek

Thank You !

Strong foundations enhance developments in formal methods.

Coalgebraic generalisations may give strong foundations.

Two recent cases in favour of bisimilarity.