UNIVERSALLY COHERENT SEMANTICS OF TYPE THEORIES

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Categorical semantics of type theories often interpret contexts and type formers using universal properties. A quintessential example is the simply-typed lambda calculus (STLC) interpreted in cartesian closed categories (CCCs). Contexts are interpreted using products and function types using exponentials. Another example is Martin-Löf type theory with extensional identity types (eMLTT) interpreted using comprehension categories with unit (CompCU). Intuitively, any universal arrow satisfying the required property is as good as any other. However, defining an honest function from syntax to semantics requires a choice for every type. For example, consider the STLC application $\Gamma \vdash tu : B$ composed of $\Gamma \vdash t : A \to B$ and $\Gamma \vdash u : A$. The CCC semantics $[\![tu]\!] : [\![\Gamma]\!] \to [\![B]\!]$ of tu requires a choice of exponential $[\![A]\!] \Rightarrow [\![B]\!]$ even though this object does not appear in the type of the conclusion judgement. Every choice of exponential will result in the same morphism $[\![tu]\!]$. We call a semantics *universally coherent* when it is independent of intermediate choices of universal arrows. We prove that the CompCU semantics of eMLTT is universally coherent.

Our method extends the work of Makkai [Mak96]. For a category $\mathcal{C}$, Makkai defines a *clique* A in $\mathcal{C}$ to be a tuple $(|A|, A_a, A(\cdot,\cdot))$ consisting of a non-empty indexing set $|A|$, objects $A_a \in \mathcal{C}$ for each $a \in |A|$, and a functorial assignment of morphisms $A(a_2, a_1) \colon A_{a_1} \to A_{a_2}$ for all $a_1, a_2 \in |A|$, in the sense that $A(a,a) = \mathrm{id}_a$ and $A(a_3,a_1) = A(a_3,a_2) \cdot A(a_2,a_1)$ for all $a, a_1, a_2, a_3 \in |A|$. Thus, each $A(a_2,a_1)$ is an isomorphism, which we call a *canonical isomorphism*. Therefore, the clique A is a collection of canonically isomorphic objects. A *morphism* $f \colon A \to B$ of cliques in a category is a family of arrows $f(b,a) \colon A_a \to B_b$ for each $a \in |A|$ and $b \in |B|$ such that $B(b_2,b_1) \cdot f(b_1,a_1) = f(b_2,a_2) \cdot A(a_2,a_1)$ for all $a_1,a_2 \in |A|, b_1,b_2 \in |B|$. These provide the category $\mathsf{C}(\mathcal{C})$ of cliques. The construction $\mathsf{C}(-) \colon \mathbf{Cat} \to \mathbf{Cat}$ is a strict pseudo-functor, and is furthermore a pseudo-monad. The Kleisli bicategory $\mathbf{Cat}_{\mathsf{C}(-)}$ is that of Makkai's *anafunctors*.

Makkai showed that for a small category $\mathcal{C}$ with finite products and exponentials, the category $\mathsf{C}(\mathcal{C})$ has a *specified* CCC structure. That is, the mere existence of the CCC structure on $\mathcal{C}$ induces a chosen structure on $\mathsf{C}(\mathcal{C})$. For $A, B \in \mathsf{C}(\mathcal{C})$ the chosen product $A \times B$ is given by:

$$|A \times B| := \{(a, b, \langle P, \pi_1, \pi_2\rangle)\, \langle P, \pi_1, \pi_2\rangle \text{ is a product of } A_a \text{ and } B_b\}$$
$$(A \times B)_x := P^x \text{ for } x := (a^x, b^x, \langle P^x, \pi_1^x, \pi_2^x\rangle)$$
$$(A \times B)(y, x) := \langle \pi_1^x, \pi_2^x\rangle^y =: \phi_{y,x},$$

which is a clique as the maps $\phi_{y,x} \colon P^x \to P^y$ are the ones induced by the universal property of being a product. The indexing set consists of all choices of product in $\mathcal{C}$, thereby avoiding a choice in $\mathcal{C}$. **Contribution A:** the $\mathsf{C}(\mathcal{C})$-valued STLC semantics amounts to universal coherence for $\mathcal{C}$. **Contribution B:** the clique-valued eMLTT semantics exists and amounts to universally coherent CompCU-semantics. See the summary of results for more detail if desired.

References

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Summary of results

Our main contributions extends this result to eMLTT. The CompCU semantics interprets context extension, substitution, and all types (i.e. 1, Π, Σ, Id) using universal properties. Let $p \colon \mathcal{E} \to \mathcal{B}$ be an *unspecified* CompCU, meaning a Grothendieck fibration p such that the universal arrows underlying the adjoints $p \dashv_{\text{fib}} 1 \colon \mathcal{B} \to \mathcal{E}$ and $1 \dashv \mathcal{P}_0 \colon \mathcal{E} \to \mathcal{B}$ exist but remain unchosen. Naively, one may expect $\mathsf{C}(p) \colon \mathsf{C}(\mathcal{E}) \to \mathsf{C}(\mathcal{B})$ to suffice, however arbitrary choices remain when attempting to specify an analogous choice-free cleavage. Thus, we first define a new functor $\mathsf{V}(p) \colon \mathsf{V}(\mathcal{E}) \to \mathcal{B}$ which is essentially $\mathsf{C}(-)$ applied fibrewise. This construction is best understood bicategorically: the Grothendieck fibration gives a pseudo-functor $\overline{p} : \mathcal{B}^{\text{op}} \to \mathbf{Cat}_{\mathsf{C}(-)}$, and the indexed category of $\mathsf{V}(p)$ post-composes $\overline{p}$ with the right adjoint $\mathbf{Cat}_{\mathsf{C}(-)} \to \mathbf{Cat}$ in the Kleisli resolution of $\mathsf{C}(-)$.

The functor $\mathsf{V}(p)$ has a specified cleavage with a specified choice of fibred terminal objects, but its comprehension structure is still unspecified. Applying the clique construction provides a cleavage of $\mathsf{C}(\mathsf{V}(p)) \colon \mathsf{C}(\mathsf{V}(\mathcal{E})) \to \mathsf{C}(\mathcal{B})$. **Contribution 1:** this cloven fibration has specified functors $1^{\mathsf{CV}} \colon \mathsf{C}(\mathcal{B}) \to$

$\mathsf{C}(\mathsf{V}(\mathcal{E}))$ and $\mathcal{P}_0^{\mathsf{CV}}\colon \mathsf{C}(\mathsf{V}(\mathcal{E})) \to \mathsf{C}(\mathcal{B})$—a CompCU structure for $\mathsf{C}(\mathsf{V}(p))$. **Contributions 2, 3, and 4:** if dependent products, dependent sums, or extensional identity types merely exist for p, then correspondingly $\mathsf{C}(\mathsf{V}(p))$ has specified dependent products, sums, or extensional identity types. **Contribution B:** the $\mathsf{C}(\mathsf{V}(p))$-valued eMLTT semantics amounts to universally coherent p-semantics.

Prospects. Our proofs that unchosen p-structure induces chosen $\mathsf{C}(\mathsf{V}(p))$-structure are linked via a general construction of cliques from fibred adjunctions. Thus, we provide a recipe for extensions to eMLTT with other features based on universal properties, such as fibred monads and fibred initial algebras. We currently investigate a weaker form of universal coherence for intensional identity types.

The proof of universal coherence uses the standard coherence theorem of Curien [Cur93] extended to comprehension categories [CGH14; LW15]. In the case of eMLTT, the standard coherence theorem concerns derivations differing by uses of coherence of transports, each of which is interpreted as a morphism induced by a universal property. We conjecture that (a) universal coherence can be proved directly without standard coherence and that (b) universal coherence implies standard coherence for eMLTT. If both conjectures are true, then our recipe for constructing the cliques needed for universal coherence may also help prove coherence for other type theories with CompCU-based semantics.

We want to present these results at SSTT to identify further directions and connections to existing relevant work.