# GENERALIZED INVERSE DIAGRAMS IN TRIBES

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Given a fibration category $\mathcal{F}$ and a direct category $C$, there is a notion of Reedy fibrant diagram $C \to \mathcal{F}$ and a corresponding notion of Reedy fibration between such diagrams, providing a fibration category structure on the category of such diagrams. If $C$ is not assumed to be a direct category, it is still possible to endow the category $\mathcal{F}^C$ with the structure of a fibration category (see [Rad06, Theorem 9.5.5]).

For tribes, which provide a categorical framework for models of type theory, the situation is different: it seems that a tribe structure on a category of diagrams $\mathcal{T}^C$, where $\mathcal{T}$ is a tribe, has only been considered in the literature when $C$ is an inverse category, in which case the notion of Reedy fibrancy can be used (see [KS19, Lemma 2.22]). However, some indexing categories of interest in the context of type theory include categories of semi-cubes, which corresponds to the notion of generalized Reedy category, a notion aiming at relaxing the definition of a Reedy category in order to allow non-identity isomorphisms (cf. [Cis+06] and [BM11]).

In this work ([Che26]), we provide a tribe structure on a subcategory of fibrant diagrams $\mathcal{T}^R$, assuming $R$ to be a generalized inverse category satisfying some mild condition (categories of symmetric semi-cubes being a motivating example). Essentially, we have the following:

**Lemma 1.** *There exists an absolutely dense functor $p : \mathbf{D}_R \to R$ from a (strictly) direct category (i.e., the precomposition functor $p^* : \mathbf{Set}^R \to \mathbf{Set}^{\mathbf{D}_R}$ is fully faithful).*

**Definition 2.** A $p$-fibrations is defined to be a morphism $m : F \to F'$ in $\mathcal{T}^{R^{op}}$ such that $p^*m$ is a Reedy fibration in $\mathcal{T}^{\mathbf{D}_R^{op}}$. We consider the full subcategory $\mathcal{T}_f^{R^{op}} \subset \mathcal{T}^{R^{op}}$ of $p$-fibrant diagrams, equipped with the $p$-fibrations.

**Theorem 3.** *$\mathcal{T}_f^{R^{op}}$ enjoys the structure of a tribe.*

## References


[BM11] Clemens Berger and Ieke Moerdijk. "On an extension of the notion of Reedy category". In: *Mathematische Zeitschrift* 269.3 (2011), pp. 977–1004.

[Che26] El Mehdi Cherradi. "Generalized Reedy diagrams in tribes". In: *arXiv preprint arXiv:2602.17355* (2026).

[Cis+06] Denis-Charles Cisinski et al. *Les préfaisceaux comme modèles des types d'homotopie.* Société mathématique de France, 2006.

[KS19] Krzysztof Kapulkin and Karol Szumiło. "Internal languages of finitely complete $(\infty, 1)$-categories". In: *Selecta Mathematica* 25.2 (2019), pp. 1–46.

[Rad06] Andrei Radulescu-Banu. "Cofibrations in homotopy theory". In: *arXiv preprint math/0610009* (2006).