Direct initial-algebra semantics for (multimodal) dependent type theories

André Hirschowitz Tom Hirschowitz Ambroise Lafont

This work in progress is about specifying dependent type theories, in the spirit of *initial-algebra semantics* [7, 6], which consists in defining the desired type theory implicitly (up to isomorphism) as the initial object in a suitable category of "models". Most concrete implementations of this idea proceed by introducing a formal notion of presentation, and associating to each presentation a category of models – with at least an initial object. Such notions of presentation include, e.g., Cartmell's [3] *generalised algebraic theories* (GATs) and Uemura's [14] *second-order algebraic theories* (SOGATs). Users of such frameworks must thus keep in mind the link between presentations and their semantics.

In this work, we explore a *direct* approach to initial-algebra semantics. This means (1) delineating an algebraic structure whose objects include potential categories of models of dependent type theories, and (2) constructing the desired objects algebraically.

We instantiate this idea for constructing 1-categories of models of type theories. Our algebraic structure is the large 2-category **RLPCAT** of locally presentable categories, right adjoints, and all natural transformations.

As others [14, 9, 4, 5, 12], our approach enables (1) building substitution into operation arities, and (2) specifying what Coraglia and Di Liberti call *extensional type constructors* [4, §3.7], a compact way of axiomatising some type formers, along with their constructors and eliminators. However, unlike these approaches, we implement initial-algebra semantics in the 1-categorical sense. We view this as a gain both (1) in accessibility, since we mostly use old-school category theory, and (2) in precision, since we implicitly define the desired type theories up to isomorphism rather than equivalence. In counterpart, model morphisms must preserve operations strictly. We illustrate our approach by designing a semantics to SOGATs, and by reconstructing the category of models of multimodal type theory [8], which SOGATs typically cannot handle.

In order to construct the desired locally presentable categories, we use mostly known 2-categorical properties of **RLPCAT**. The main idea is to define categories of models by pullback along suitable "refinement" functors. Below left are a few typical refinement functors, where $\rightarrow$, $\mathbb{R}$, $\mathbb{S}^1$, **adj**, and $\cong$ respectively denote the free-standing arrow, retraction, parallel pair of arrows, isomorphism, and adjunction[1].

$$\begin{array}{ccccc} \mathbf{Set}^{\rightarrow} & \mathbf{Set}^{\mathbb{R}} & \mathbf{Cat}^{\rightarrow} & \mathbf{Cat}^{\cong} & \mathbf{Cat}^{\mathbf{adj}} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{Set}^{2} & \mathbf{Set}^{\rightarrow} & \mathbf{Cat}^{\mathbb{S}^1} & \mathbf{Cat}^{\rightarrow} & \mathbf{Cat}^{\rightarrow} \end{array} \qquad \begin{array}{ccc} \bullet & \longrightarrow & \mathbf{Set}^{\rightarrow} \\ \downarrow \lrcorner & & \downarrow{\scriptstyle \text{refinement}} \\ \mathcal{A} & \xrightarrow[\text{arity}]{} & \mathbf{Set}^{2} \end{array}$$

For a given category $\mathcal{A}$ of models, one defines an "arity" functor to the codomain of a suitable refinement functor, and takes the pullback. Typically, an arity functor $\mathcal{A} \to \mathbf{Set}^2$ picks two sets $X(A)$ and $Y(A)$ computed from each model A; and the pullback (above right) consists of models M equipped with a map $X(A) \to Y(A)$. Similarly, the other refinements respectively equip models with a section, an equation, an isomorphism, and a right adjoint.

The main challenge, then, is to be able to construct relevant refinement and arity functors 2-categorically within **RLPCAT**. Our main tools are Bird's *flexible limits* [1] and 2-categorical notions (op)fibrations [13]. One well-known technical aspect that is specific to 1-categorical models is that, in order for our pullbacks to remain in **RLPCAT**, we need to ensure that refinement functors to be isofibrations [10, 2]. For comparison, Uemura and Nguyen [11] use ∞-pullbacks for related purposes, which lifts this constraint.

[1]Beware, $\mathbf{Cat}^{\mathbf{adj}}$ denotes the 1-category of 2-functors $\mathbf{adj} \to \mathbf{Cat}$, 2-natural transformations.

References

[1] G.J. Bird. *Limits in 2-Categories of Locally-Presented Categories.* PhD thesis, University of Sydney, 1984.

[2] John Bourke and Richard Garner. Monads and theories. *Advances in Mathematics*, 351:1024–1071, July 2019.

[3] John Cartmell. Generalised algebraic theories and contextual categories. *Annals of Pure and Applied LogicAnn. Pure and Appl. Logic*, 32:209–243, 1986.

[4] Greta Coraglia and Ivan Di Liberti. Context, judgement, deduction. *CoRR*, abs/2111.09438, 2021.

[5] Ivan Di Liberti and Axel Osmond. Bi-accessible and Bipresentable 2-Categories. *Applied Categorical Structures*, 33(1):3, February 2025.

[6] J.A. Goguen, J.W. Thatcher, and E.G. Wagner. An initial algebra approach to the specification, correctness and implementation of abstract data types. In R. Yeh, editor, *Current Trends in Programming Methodology, IV: Data Structuring*, pages 80–144. Prentice-Hall, 1978.

[7] Joseph A. Goguen and James W. Thatcher. Initial algebra semantics. In *15th Annual Symposium on Switching and Automata Theory (SWAT)*, pages 63–77. IEEE, 1974.

[8] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. Multimodal Dependent Type Theory. *Logical Methods in Computer Science*, Volume 17, Issue 3:7571, July 2021.

[9] Daniel Gratzer and Jonathan Sterling. Syntactic categories for dependent type theory: Sketching and adequacy. *CoRR*, abs/2012.10783, 2020.

[10] André Joyal and Ross Street. Pullbacks equivalent to pseudopullbacks. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, XXXIV(2):153–156, 1993.

[11] Hoang Kim Nguyen and Taichi Uemura. ∞-type theories. *Higher Structures*, 9(1), 2025.

[12] Jonathan Osser. A 2-sketchy approach to type theory. Master's thesis, Universiteit van Amsterdam, 2025.

[13] Ross Street. Fibrations and Yoneda's lemma in a 2-category. In Gregory M. Kelly, editor, *Category Seminar*, volume 420, pages 104–133. Springer Berlin Heidelberg, Berlin, Heidelberg, 1974.

[14] Taichi Uemura. A general framework for the semantics of type theory. *Mathematical Structures in Computer Science*, 33(3):134–179, 2023.