TAMING REVERSALS IN CUBICAL TYPE THEORIES

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Cubical type theories extend Martin-Löf type theory with an abstract interval $\mathbb{I}$. Types of *paths* $a_0 \sim^A a_1$, i.e., of terms $i : \mathbb{I} \vdash p(i) : A$ varying over the interval with $p(0) = a_0$ and $p(1) = a_1$, play the role of equality types. The range of strict equations enjoyed by path types depends on the *interval theory*, the operations on $\mathbb{I}$. With a *reversal* $i : \mathbb{I} \vdash \neg i : \mathbb{I}$ such that $\neg 0 = 1$, $\neg 1 = 0$, and $\neg\neg i = i$, we get a strictly involutive symmetry $\mathsf{sym} : (a_0 \sim^A a_1) \to (a_1 \sim^A a_0)$. Without a reversal, symmetry is definable using *Kan filling*, but is only involutive up to a path. Cubical Agda, the most widely used cubical proof assistant, includes $\neg$ as well as *connections* $\vee$ and $\wedge$, binary operators on $\mathbb{I}$ mimicking topological max and min.

While convenient, a rich interval theory is harder to justify. We do not know if extending the interval theory is in general *conservative*: that besides making some equations strict, it does not change what is provable. For a cubical type theory to be a language for algebraic topology, it should have a model in ∞-*groupoids*, i.e., “spaces”. Such a model exists for the theory with $\vee$, and the second author recently announced a model for the theory of a distributive lattice $(\mathbb{I}, \wedge, \vee, 0, 1)$. However, neither models a reversal.

We present an argument that a reversal is a harmless extension to cubical type theory using a “twist structure” construction from lattice theory. When $\mathbb{I}$ is an interval, its square $\mathbb{I} \times \mathbb{I}$ is an interval with endpoints $(0, 1)$ and $(1, 0)$ and a reversal $\neg(i_0, i_1) \coloneqq (i_1, i_0)$. For a distributive lattice $(\mathbb{I}, \wedge, \vee, 0, 1)$, the square is a De Morgan algebra with $(i_0, i_1) \wedge (j_0, j_1) \coloneqq (i_0 \wedge j_0, i_1 \vee j_1)$ and $(i_0, i_1) \vee (j_0, j_1) \coloneqq (i_0 \vee j_0, i_1 \wedge j_1)$. In general, when $\mathbb{I}$ has some self-dual algebraic structure, $\mathbb{I} \times \mathbb{I}$ has the same structure and a reversal.

Using this observation, we prove two results. The first applies to cubical type theories where certain strict equations for the Kan filling operator and higher inductive types are weakened. In this setting, we prove that adding a reversal to a cubical type theory with a self-dual interval theory is conservative using a “twist translation” back to the base theory.

Our second result is for standard cubical type theories. Given a model without reversals of a form specified by Angiuli, Brunerie, Coquand, Harper, Favonia, and Licata, we build a model with a reversal whose interval as the square of the original model's interval. The new model has the same homotopy theory as the original; applying the construction to a model in ∞-groupoids, we get a model of reversals in ∞-groupoids. We plan to adapt this construction to the model announced by the second author to obtain a model in ∞-groupoids with Cubical Agda's interval theory.