INVERSE DIAGRAM MODELS IN CWFS VIA PARAMETRICITY

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Reedy-fibrant inverse diagram models were first introduced by Shulman in the setting of *type-theoretic fibration categories* [Shu15], and were subsequently developed by Kapulkin and Lumsdaine in the more syntactically strict setting of *categories with attributes* [KL21]. Despite their long history, they form a comparatively small corner of the literature on the semantics of dependent type theories.

Our work on *Displayed Type Theory (dTT)* [KS25] belongs to the modern tradition of constructive univalent models and computational adequacy. However, our cube category of interest, unary semi-cubes, lacks degeneracies and is direct. This leads us to study Reedy-fibrant presheaves valued in an arbitrary *category with families*, rather than only Set-valued presheaves. dTT reifies the backwards-shift operation resembling path-space formation

$$\left(- \otimes \square^1\right)^* : \mathcal{C}^{\boxed{1}^{\mathrm{op}}} \to \mathcal{C}^{\boxed{1}^{\mathrm{op}}}$$

by way of an internal parametricity operation $(-)^{\mathsf{d}}$, called *display*, whose computation laws hold definitionally. For example, display sends Π-types to iterated Π-types. The type of functions *displayed over* a function f is:

$$\left((x : A) \to B\ [x]\right)^{\mathsf{d}}\ f \equiv (x : A)\,(x' : A^{\mathsf{d}}\ x) \to B^{\mathsf{d}}\ [x, x']\ (f\ x)$$

This talk describes a construction that, from any input CwF $\mathcal{C}$ with Π-types and universes, produces a CwF $\mathcal{C}^{\boxed{1}^{\mathrm{op}}}$ of Reedy-fibrant presheaves on the unary semi-cube category. The construction is carried out levelwise by direct parametricity formulas: the Reedy CwF structure, including Π-types and universes, is defined explicitly rather than obtained by subsequent coherence results. As a consequence, the computation laws for display hold strictly on the nose.

This gives a hands-on constructive account of inverse diagram models in a CwF setting tailored to the case of nondegenerate unary semi-cubes.

REFERENCES

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