ELEMENTARY ∞-TOPOSES FROM TYPE THEORY

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Since its conception, it has been speculated that Homotopy Type Theory (HoTT) [Uni13] is the internal language of particular higher categories also called *elementary ∞-toposes*. A precise formulation of this statement was given by Kapulkin and Lumsdaine in [KL18, Conj. 3.7] and is known as the *internal language conjecture*. Here, HoTT is understood as Martin-Löf dependent type theory with dependent sums, dependent products, intensional identity types and univalent universes.

Proving a syntax-semantics correspondence between HoTT and elementary ∞-toposes is currently an open problem. Elementary ∞-toposes are supposed to generalise both Grothendieck ∞-toposes and ordinary 1-toposes. We define an elementary ∞-topos as a finitely complete, locally cartesian closed ∞-category with enough univalent morphisms. In its precise statement, the internal language conjecture then asserts that there is a Dwyer-Kan equivalence $\mathrm{Cl}_\infty \colon \mathbf{CxlCat}_{\mathtt{HoTT}} \xrightarrow{\sim} \mathbf{ElTop}_\infty$ between the category of categorical models of HoTT and the category of elementary ∞-toposes, induced by sending each model to its ∞-localisation at the class of homotopy equivalences. Proving that this functor exists, i.e. that the ∞-localisation takes values in $\mathbf{ElTop}_\infty$, and showing that this is a Dwyer-Kan equivalence has so far been an open problem.

In the talk, we will present the work [AP25] in which we prove the existence of this functor Cl_∞, a first step towards proving the conjecture, as well as ongoing work in which this result is generalised and where we consider also higher inductive types and colimit types. Concretely, we show that every model of HoTT presents an elementary ∞-topos via its ∞-localisation. First, we use the fact that every model of HoTT has the structure of a tribe in the sense of Joyal [Joy17]. We extend Joyal's theory of tribes by introducing the notion of a univalent fibration in a tribe and the notion of a univalent tribe and we show that every categorical model of HoTT is such a univalent tribe. Then, we prove that every univalent tribe presents via its ∞-localisation an elementary ∞-topos. Thus, the functor Cl_∞ can be obtained as a composite: $\mathbf{CxlCat}_{\mathtt{HoTT}} \to \mathbf{UnivTrb} \to \mathbf{ElTop}_\infty$.

We also show that our definition of elementary ∞-topos implies the existence of a small subobject classifier for every univalent morphism p classifying those monomorphisms that are classified by p.

We finally explain some ongoing work about proving that the existence of pushout types and empty types implies the existence of finite colimits upon applying Cl_∞ and discuss future directions.

References

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