

## COMPLETENESS FOR DISPLAY CATEGORIES

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The notion of a category of a pullback-stable class of “display maps” was introduced in my 1986 PhD thesis but is best treated in Chapter VIII of my book *Practical Foundations of Mathematics* (CUP 1999). However, the book made a mess of proving that every display category is weakly equivalent to the category of contexts and substitutions of some generalised algebraic theory. That proof is the main purpose of this paper. We do not assume that the display maps have identities, composition or adjoints, which correspond to unit, sum and quantifier types.

There is a *very simple* construction of this category *via* a sketch, in contrast to the *very complicated* recursion that is required to do it directly. A sketch is a system of generators and relations for a category, for us *without* specified limit cones or colimit cocones.

Ours corresponds very closely to the syntax of the theory. Its objects are the contexts. There are two classes of generating morphisms: one has the display or weakening maps for the types in context and the other consists of the typed terms in context. The standard equations are the famous substitution lemma and its simpler analogues for weakening. The particular theory may add further equations.

The book shows that each syntactic morphism has a normal form as a sequence of substitutions of terms for variables followed by forgetting the old variables. It also explicitly proves that substitution is pullback.

Here we first show that any category without *specified* pullbacks is weakly equivalent to one with them. This proliferates the display maps by introducing a new one for each pullback situation, resulting in a *lot* of copies of the objects. This gives essential surjectivity, so long as each object has a sequence of displays to the terminal object.

For fullness, we must express any morphism between the sources of sequences of display maps in the same pattern as the normal form for syntactic morphisms. The key to this is being able to “nibble off” parts of the morphism corresponding to terms. Display maps do this at one end, whilst a certain pullback mediator does it at the other end.

For faithfulness, each splitting of a display map gives a function-symbol, but there must be an equation to say that substituting into one symbol is equal to the symbol given by the pullback.