Path Types in Algebraic Type Theory (Abstract)

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We propose a new approach to the semantics of identity types in intensional Martin-Löf type theory (MLTT), assuming only a category with finite limits and a bipointed, exponentiable object (an *interval*). The approach therefore applies in many different settings such as clans [1] and natural models [2]. It is adopted in the HoTTLean project [3], and has been formalized.

The specification of *extensional* identity types in [2] parallels that of the other type formers Σ and Π, but the treatment of the *intensional* case there was less uniform. Here an improvement is achieved by employing an interval in order to give a single pullback specification, in the style of the other type-formers, of a model with *path types*. The interval is also used to specify a (Hurewicz) fibration structure on the universe of the model. It is shown that the combination of these two conditions already suffices to model the intensional identity rules, assuming only finite limits. The addition of an interval also relates the current treatment to that of cubical type theory [4].

Let $\mathcal{E}$ be a finite limit category with an interval $1 \rightrightarrows \mathbf{I}$. For every object A, there is then a *path object factorization* of the diagonal $\delta : A \to A \times A$, obtained by exponentiating A by $1 \rightrightarrows \mathbf{I} \to 1$,

$$A \xrightarrow{\rho} A^{\mathbf{I}} \xrightarrow{\varepsilon} A \times A.$$

Definition 1. A natural model $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ in a finite limit category $\mathcal{E}$ with an interval will be said to have *path types* if there are structure maps $(\mathsf{Path}, \mathsf{path})$ making a pullback square,

$$\begin{array}{ccc} \dot{\mathsf{T}}^{\mathbf{I}} & \xrightarrow{\mathsf{path}} & \dot{\mathsf{T}} \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\mathsf{t}} \\ \dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}} & \xrightarrow[\mathsf{Path}]{} & \mathsf{T} \end{array} \tag{A1}$$

where $\varepsilon : \dot{\mathsf{T}}^{\mathbf{I}} \to \dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}}$ is the relative pathobject of $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ over T. We also require the map $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ to be a *Hurewicz fibration*, in the usual sense of path-lifting.

The usual Id-Elim rule now holds for any type family $A \to X$ classified by $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$. Indeed, $A \to X$ is Hurewicz since it is a pullback of $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$. The path type $A^{\mathbf{I}} \to A \times_X A$ is also Hurewicz, since it is a pullback of $\dot{\mathsf{T}}^{\mathbf{I}} \to \dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}}$, which is Hurewicz by (A1). It follows that $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ has a *connection* $\chi : \dot{\mathsf{T}}^{\mathbf{I}} \to (\dot{\mathsf{T}}^{\mathbf{I}})^{\mathbf{I}} \cong \dot{\mathsf{T}}^{\mathbf{I}\times\mathbf{I}}$ (over T), and therefore so do all of its pullbacks. One then shows that the connection, together with path-lifting, suffices for the usual rule of Id-Elim.

Proposition 2. *Let* $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ *be a natural model in a finite limit category* $\mathcal{E}$ *with an interval* $1 \rightrightarrows \mathbf{I}$*. If* $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ *has path types* (A1)*, and is a (normal) Hurewicz fibration, then for all classified types* $A \to X$*, the path type* $A \to A^{\mathbf{I}} \to A \times_X A$ *satisfies the identity rules of intensional MLTT.*

While not every model of MLTT admits path types, many of the naturally occuring ones do, including the groupoid, cubical, and simplicial models. And when $\mathbf{I}$ is the terminal object, all maps are Hurewicz fibrations, and we recover the semantics of extensional identity types from [2].

References

[1] André Joyal. Notes on clans and tribes. Unpublished notes, *arXiv:1710.10238*, 2017.
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[4] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom. *TYPES 2015*.