The category of nominal sets is locally Clouston closed

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Abstract

The category **Nom** of nominal sets was proposed by Pitts and Gabbay as a setting for the semantics of abstract syntax with variable bindings. It is well-known that **Nom** is a topos, also known as the Schanuel topos, and in particular it follows that **Nom** is *locally closed*, i.e., every slice category $\mathbf{Nom}/X$ is cartesian-closed. In this paper, we show that **Nom** has a much stronger property: every monoidal (closed) structure on **Nom** induces a monoidal (closed) structure on all slice categories $\mathbf{Nom}/X$. One monoidal closed structure of particular interest on **Nom** is the *separated product* $A * B$, whose right adjoint $A \multimap\!\!* \; B$ we call the *Clouston function space*. In particular, it follows that **Nom** is locally Clouston closed.

Keywords: Nominal sets, monoidal closed categories, local closure.

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1 Introduction

The category **Nom** of nominal sets was proposed by Pitts and Gabbay [7,8] as a setting for the semantics of abstract syntax with variable bindings. This category has a wealth of interesting structure. Among other things, it has all limits and colimits, it is cartesian closed, and in fact a topos (in this context, it is also known as the *Schanuel topos*). But the feature that makes **Nom** uniquely interesting is the presence of *name abstraction*: for every object X (whose elements we may think of as the *terms* of some syntax), there is an object $[\mathbb{A}]X$, whose elements represent *abstractions*, which we can think of as terms $a.t$ with a bound variable a. Crucially, the elements of $[\mathbb{A}]X$ are automatically defined up to α-equivalence, i.e., up to renaming of bound variables. This makes **Nom** suitable as a category in which to interpret abstract data types with binders. For example, the nominal set T of lambda terms can be defined as the initial fixed point of the equation $T = \mathbb{A} + T \times T + [\mathbb{A}]T$, reflecting the fact that the set of terms is freely generated so that every term is either a variable ($\mathbb{A}$), an application ($T \times T$), or a lambda abstraction ($[\mathbb{A}]T$). The fact that this simple definition automatically takes care of bound variables and α-equivalence is what makes nominal sets so useful in this context.

Theorem provers such as Rocq, Agda, and Lean are based on dependent type theory [14,1,5]. Seely showed that an appropriate categorical structure for giving semantics of dependent type theories is a

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locally cartesian closed category, i.e., a category $\mathbf{C}$ such that all of its slice categories $\mathbf{C}/X$ are cartesian closed [15] (see also [3]). Since **Nom** is locally cartesian closed, one might imagine that it can support the semantics of a dependent type theory in which one can also reason about data types with binders [12]. However, defining such a type theory has so far proved to be elusive; some steps in this direction were taken by [2,13,6,16]. One of several issues that need to be addressed is that in such a dependent type theory, it is not sufficient to have a global object of name abstractions; instead, we must consider kinding judgements such as $\Gamma \vdash [\mathbb{A}]X : \mathrm{type}$, where X depends on variables defined in the context Γ. As is usual in dependent type theory, we will interpret a kinding judgement $\Gamma \vdash X : \mathrm{type}$ as a map $X \to \Gamma$, i.e., an object in $\mathbf{Nom}/\Gamma$. Given such an object, we then need to define the interpretation of $[\mathbb{A}]X$, also as an object in $\mathbf{Nom}/\Gamma$. Therefore, not just **Nom** itself, but also all of its slice categories $\mathbf{Nom}/\Gamma$ should support name abstractions and other related constructs.

In this paper, we begin to address this issue by showing that the required local structure indeed exists in **Nom**. In fact, we show something more general. Clouston showed that name abstraction $[\mathbb{A}]X$ arises as a special case of a more general monoidal closed structure on **Nom**, given by the so-called *separated product* $*$ and its right adjoint $\multimap\!\!*$, which we call the *Clouston function space*. As the title of this paper suggests, we show that **Nom** is *locally Clouston closed*, i.e., there is a version of $*$ and $\multimap\!\!*$ on every slice category $\mathbf{Nom}/\Gamma$. However, in proving this, we found something that surprised us: indeed, one can show much more generally that *every* monoidal (respectively, monoidal closed) structure on **Nom** lifts to a monoidal (respectively, monoidal closed) structure on $\mathbf{Nom}/\Gamma$.

This is surprising because it seems to be very rare. In generic categories, not only is it not usually possible to transport monoidal structure from the base category to its slice categories, but there is not even any canonical notion of what it would mean to do so. In other words, there is no obvious way in which one can say that a monoidal structure on a slice category is “the same as” that on the base category — and consequently, there is no accepted definition of a “locally monoidal” (or “locally monoidal closed”) category. It seems to be a particular feature of **Nom** that there is such a notion, and moreover all such structures are local.

The structure of the paper is as follows: Section 2 covers the basics of nominal sets. In Section 3, we discuss how one can extend the Π-action from finitely supported permutations to all permutations. In Section 4, we define how every bijection between atom sets induces a functor between the category of nominal sets over those atom sets. In Section 5, we give the definition of nominal families over a nominal set X and prove that the category of nominal families over X and the slice category over X are equivalent. Section 6 covers the monoidal structure of the slice category $\mathbf{Nom}/X$, while Section 7 covers the monoidal closed structure of $\mathbf{Nom}/X$.

2 Background

In this section, we will review some of the basic notions of nominal set theory. For a much more detailed introduction, see [12].

2.1 Names and permutations

We fix a countably infinite set $\mathbb{A}$, whose elements will be called *atoms* or *names*. Atoms serve as unique identifiers. Their two main properties are that we can always find a previously unused (“fresh”) atom, and that they can be compared for equality.

Definition 2.1. A *finitely supported permutation* of $\mathbb{A}$ is a permutation $\pi : \mathbb{A} \to \mathbb{A}$ such that the set $\{a \in \mathbb{A} \mid \pi(a) \neq a\}$ is finite. Let $\Pi_{\mathbb{A}}$ be the group of these finitely supported permutations. We drop the subscript when it is clear from the context. The group operation is the composition of permutations, which we denote by $\pi_1 \circ \pi_2$ or simply $\pi_1\pi_2$. The identity permutation is denoted by id.

Definition 2.2. A Π-set is a pair $(X, \bullet)$, where X is a set and $\bullet$ is an action of Π on X. We often just write X for a Π-set when the action is clear from the context.

Definition 2.3. Let X and Y be Π-sets. A function $f : X \to Y$ is *equivariant* if for all $\pi \in \Pi$ and for all $x \in X, \pi \bullet (f(x)) = f(\pi \bullet x)$.

Definition 2.4. Let X be a Π-set. Then the action of Π on $\mathcal{P}(X)$, the powerset of X, is defined by

$\pi \bullet S := \{\pi a \mid a \in S\}$. This means that $\mathcal{P}(X)$ is a Π-set.

Definition 2.5. Let X and Y be Π-sets and let $\pi \in \Pi$. The action of π on a function $f : X \to Y$ is defined by $(\pi \bullet f)(x) = \pi \bullet \left(f\left(\pi^{-1} \bullet x\right)\right)$.

2.2 Support

We write $X - Y$ to denote the set difference, so $X - Y = \{x \in X \mid x \notin Y\}$.

Definition 2.6. Let X be a Π-set and let $x \in X$. We say that the set $S \subseteq \mathbb{A}$ is a *support* for x if for all $a, b \in \mathbb{A} - S$, $(a\,b) \bullet x = x$. Given a Π-set X, $x \in X$ is *finitely supported* if there exists some finite support S for it. One can show that for any finitely supported x, there exists a smallest finite support, and we denote it by $\mathrm{supp}(x) \in P(\mathbb{A})$ [12].

Definition 2.7. A Π-set X is called a *nominal set* if all $x \in X$ are finitely supported.

Every set is nominal with the trivial action ("discrete" nominal set). One non-trivial example of a nominal set is $\mathbb{A}$ itself, with the natural action $\pi \bullet a = \pi(a)$. Moreover, if X, Y are nominal sets, then the set of finitely supported functions $f : X \to Y$ is also a nominal set, with the action as in Definition 2.5. Also, if $f : X \to Y$ is finitely supported, then for all $x \in X$, we have $\mathrm{supp}(f(x)) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(x)$. The function f is equivariant if and only if $\mathrm{supp}(f)$ is empty; in this case, we have $\mathrm{supp}(f(x)) \subseteq \mathrm{supp}(x)$. We will discuss other examples of nominal sets below.

2.3 Freshness

Let X and Y be nominal sets and consider $x \in X$ and $y \in Y$. We say x is *fresh for* y, denoted $x\#y$, if they have disjoint supports, i.e., if $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$. Note that for all $\pi \in \Pi$, we have $x\#y \Rightarrow (\pi \bullet x)\#(\pi \bullet y)$. In other words, the freshness relation is equivariant.

Now consider a finitely supported function $P : \mathbb{A} \to \mathbf{Bool}$, which we think of as a property of elements of $\mathbb{A}$. One can show that $P(a)$ holds for some $a\#P$ if and only if $P(a)$ holds for all $a\#P$ if and only if $P(a)$ holds for all but finitely many $a \in \mathbb{A}$. This motivates the *freshness quantifier*: we are justified in saying "for fresh a" instead of "for all fresh a" or "for some fresh a".

2.4 The category **Nom**

Fix an atom set $\mathbb{A}$. The category $\mathbf{Nom}_{\mathbb{A}}$ has nominal sets as objects and equivariant functions as morphisms. We omit the subscript when it is clear from the context.

The category **Nom** has arbitrary coproducts, which are given by disjoint union with the obvious Π-action. It also has finite products, where the Π-action on $X \times Y$ is given by $\pi \bullet (x, y) = (\pi \bullet x, \pi \bullet y)$. It also has infinite products, but they are not the same as in **Set**: we must consider the subset of the infinite cartesian product that consists of finitely supported tuples. The category **Nom** is also cartesian closed, where the nominal function space $X \to_{\mathrm{nom}} Y$ is the set of all finitely supported functions $f : X \to Y$. In addition, **Nom** has many other interesting properties; for example, it is a topos.

2.5 Clouston closure

In this paper, we are particularly interested in monoidal closed structures on **Nom**. Besides the cartesian closed structure, there is one other monoidal closed structure that is of particular interest to us: the separated product and Clouston function space. Given two nominal sets X, Y, their *separated product* is defined as $X * Y = \{(x, y) \mid x \in X,\ y \in Y,\ x\#y\}$. One can show that the separated product defines a monoidal structure on **Nom**. Moreover, it has a right adjoint $-\!\!*$, which we call the *Clouston closed structure* since it is due to Clouston [4]. Its elementwise definition is a bit complicated, but we will not need it here. See [4, Def. 3.2] for the details.

It is worth mentioning that a special case of Clouston closure is *atom abstraction*: For every nominal set X, there is a nominal set $\mathbb{A} -\!\!* X$, sometimes also written $[\mathbb{A}]X$, whose elements are pairs (a, x) up to α-equivalence: here (a, x) is equivalent to (b, y) if for fresh c, $(a\,c) \bullet x = (b\,c) \bullet y$. An equivalence class of such pairs is usually written $a.x$ or $\langle a \rangle x$, and represents a *binder*. This is what is used to model "syntax with binders", and is a large part of what makes the category **Nom** interesting.

3 Extended Π action

Recall that Π is the group of finitely supported permutations of $\mathbb{A}$. Let Π_{large} be the group of all permutations of $\mathbb{A}$ (not necessarily finitely supported). If X is a nominal set, then the group Π acts on X by definition. Perhaps surprisingly, there is also a canonical action of Π_{large} on X [9,11]. The following lemma is instrumental in proving this.

Lemma 3.1. *Let X be a nominal set and let $x \in X$. Let $\pi', \pi'' \in \Pi$ be two finitely supported permutations. If $\pi'|_{\mathrm{supp}(x)} = \pi''|_{\mathrm{supp}(x)}$ then $\pi' \bullet x = \pi'' \bullet x$.*

Proof. Recall that if $\mathrm{supp}(\pi)$ is disjoint from $\mathrm{supp}(x)$, then $\pi \bullet x = x$. Now assume $\pi'|_{\mathrm{supp}(x)} = \pi''|_{\mathrm{supp}(x)}$ and consider $\pi = (\pi'')^{-1}\,\pi' \in \Pi$. Then $\pi|_{\mathrm{supp}(x)} = \mathrm{id}|_{\mathrm{supp}(x)}$, since for all $a \in \mathrm{supp}(x)$, we have $\pi'(a) = \pi''(a)$. So we have $\mathrm{supp}(\pi) \cap \mathrm{supp}(x) = \emptyset$, which implies $\pi \bullet x = x$, and hence $\pi' \bullet x = \pi'' \bullet x$, as claimed. ☐

We can now show that the Π-action on a nominal set can be extended to Π_{large}.

Proposition 3.2. *Let $(X, \bullet)$ be a nominal set. Then the action $\bullet : \Pi \times X \to X$ can be canonically extended to an action $\hat{\bullet} : \Pi_{\mathrm{large}} \times X \to X$.*

Proof. See Appendix A. ☐

We note that if σ happens to be finitely supported, then $\sigma \mathbin{\hat{\bullet}} x$ and $\sigma \bullet x$ agree.

4 Permutation Induced Functors

4.1 The functor $\bar{\iota}$ induced by injective atom renaming

In this section, we consider what happens when we move from one atom set to another. So let $\mathbb{A}$ and $\mathbb{B}$ be two atom sets and consider an injective function $\iota : \mathbb{A} \to \mathbb{B}$. We first note that this induces a group homomorphism from $\Pi_{\mathbb{A}}$ to $\Pi_{\mathbb{B}}$.

Definition 4.1. Let $\mathbb{A}, \mathbb{B}$ be atom sets and let $\iota : \mathbb{A} \to \mathbb{B}$ be an injective function. We define $\Pi_\iota : \Pi_{\mathbb{A}} \to \Pi_{\mathbb{B}}$ to be the unique group homomorphism such that $\Pi_\iota\left((a\,a')\right) = (b\,b')$ whenever $b = \iota(a)$ and $b' = \iota(a')$. Equivalently, for any permutation $\pi \in \Pi_{\mathbb{A}}$ and any atom b, we have

$$\Pi_\iota(\pi)(b) = \begin{cases} \iota\left(\pi\left(\iota^{-1}(b)\right)\right) & \text{if } b \in \mathrm{img}(\iota) \\ b & \text{otherwise.} \end{cases}$$

The above group homomorphism allows us to turn any $\Pi_{\mathbb{B}}$-action on a set X into a $\Pi_{\mathbb{A}}$-action on X by $\pi \diamond x = \Pi_\iota(\pi) \bullet x$. Indeed, this defines a functor $\bar{\iota} : \mathbf{Nom}_{\mathbb{B}} \to \mathbf{Nom}_{\mathbb{A}}$.

Definition 4.2. Given atom sets $\mathbb{A}, \mathbb{B}$ and an injective function $\iota : \mathbb{A} \to \mathbb{B}$, the functor $\bar{\iota} : \mathbf{Nom}_{\mathbb{B}} \to \mathbf{Nom}_{\mathbb{A}}$ is defined as follows.

- On objects: For $(X, \bullet) \in \mathbf{Nom}_{\mathbb{B}}$, define $\bar{\iota}(X, \bullet) = (X, \diamond)$ where $\pi \diamond x = \Pi_\iota(\pi) \bullet x$ for all $\pi \in \Pi_{\mathbb{A}}$.
- On morphisms: Given an equivariant map $f : (X, \bullet) \to (Y, \bullet)$ in $\mathbf{Nom}_{\mathbb{B}}$, we define $\bar{\iota}(f) : (X, \diamond) \to (Y, \diamond)$ in $\mathbf{Nom}_{\mathbb{A}}$ by $\bar{\iota}(f)(x) = f(x)$. In other words, $\bar{\iota}(f)$ and f have the same underlying function $X \to Y$.

To show that this is a well-defined functor, we first need to show that $(X, \diamond)$ is a nominal set. First, it is clear that $\diamond$ is an action since Π_ι is a group homomorphism. We must show that every $x \in X$ is finitely supported. We claim that $S = \{a \in \mathbb{A} \mid \iota(a) \in \mathrm{supp}_{\mathbb{B}}(x)\}$ is a support of x. Indeed, for $a', a'' \notin S$, we have $(a'a'') \diamond x = (\iota(a'), \iota(a'')) \bullet x = x$, since $\iota(a'), \iota(a'') \notin \mathrm{supp}_{\mathbb{B}}(x)$. Therefore, x is finitely supported, and $(X, \diamond)$ is a well-defined nominal set. Note that we have proved that $\mathrm{supp}_{\mathbb{A}}(x) \subseteq S$, but it is not hard to show that actually $\mathrm{supp}_{\mathbb{A}}(x) = S$.

Next, we need to show that $\bar{\iota}$ is well-defined on morphisms. So given $f : (X, \bullet) \to (Y, \bullet)$ in $\mathbf{Nom}_{\mathbb{B}}$, we must show that $\bar{\iota}(f)$ is equivariant. Therefore, consider any $\pi \in \Pi_{\mathbb{A}}$. We must show that $\bar{\iota}(f)(\pi \diamond x) = \pi \diamond \bar{\iota}(f)(x)$. Indeed, we have:

$$\pi \diamond \bar{\iota}(f)(x) = \pi \diamond f(x) = \Pi_\iota(\pi) \bullet f(x) = f(\Pi_\iota(\pi) \bullet x) = f(\pi \diamond x) = \bar{\iota}(f)(\pi \diamond x)$$

So, $\overline{\iota}(f)$ is equivariant. The functoriality of $\overline{\iota}$ is then trivial, since the composition in both categories is just composition of the underlying functions, which are the same.

4.2 The functor $\widehat{\mu}$ induced by bijective atom renaming

An important special case of Section 4.1 arises when the atom renaming function is a bijection $\mu : \mathbb{A} \to \mathbb{B}$. In this case, $\Pi_\mu : \Pi_\mathbb{A} \to \Pi_\mathbb{B}$ is an isomorphism of groups, and $\overline{\mu} : \mathbf{Nom}_\mathbb{B} \to \mathbf{Nom}_\mathbb{A}$ is an isomorphism of categories. Moreover, we have $\Pi_\mu(\pi) = \mu\pi\mu^{-1}$ and $\pi \diamond x = (\mu\pi\mu^{-1}) \bullet x$, for all $\pi \in \Pi_\mathbb{A}$.

The operation that maps $\mu : \mathbb{A} \to \mathbb{B}$ to $\overline{\mu} : \mathbf{Nom}_\mathbb{B} \to \mathbf{Nom}_\mathbb{A}$ is contravariant. Since μ is invertible, it is often more convenient to work with its inverse. We therefore define

$$\widehat{\mu} = \overline{\mu^{-1}} : \mathbf{Nom}_\mathbb{A} \to \mathbf{Nom}_\mathbb{B}.$$

Since we use the functor $\widehat{\mu} : \mathbf{Nom}_\mathbb{A} \to \mathbf{Nom}_\mathbb{B}$ a lot, it is useful to spell out its definition for the record:

Definition 4.3. For a bijection $\mu : \mathbb{A} \to \mathbb{B}$, the functor $\widehat{\mu} : \mathbf{Nom}_\mathbb{A} \to \mathbf{Nom}_\mathbb{B}$ is given as follows:

- On objects: For every $(X, \bullet) \in \mathbf{Nom}_\mathbb{A}$, we have $\widehat{\mu}(X, \bullet) = (X, \diamond)$, where
$$\pi \diamond x = (\mu^{-1}\pi\mu) \bullet x.$$
- On morphisms: For every equivariant function $f : (X, \bullet) \to (Y, \bullet)$, the morphism $\widehat{\mu}(f) : (X, \diamond) \to (Y, \diamond)$ is defined by $\widehat{\mu}(f) = f$.

Remark 4.4. The fact that $\widehat{\mu}$ acts as the identity on the underlying functions of the morphisms was already proved in Section 4.1, but it is even easier to see in the present context, because we have

$$f(\pi \diamond x) = f\big((\mu^{-1}\pi\mu) \bullet x\big) = (\mu^{-1}\pi\mu) \bullet f(x) = \pi \diamond f(x).$$

Therefore, f is equivariant with respect to both the $\bullet$-action and the $\diamond$-action.

Recall from Section 3 that every bijection $\sigma : \mathbb{A} \to \mathbb{A}$ (whether or not it is finitely supported) acts on every $\mathbb{A}$-nominal set via $\hat{\bullet}$. One may ask whether the action $\sigma \mathbin{\hat{\bullet}} (-) : X \to X$ is a natural transformation. However, this is not the case. In fact, $\sigma \mathbin{\hat{\bullet}} (-) : X \to X$ is not even a morphism of $\mathbf{Nom}_\mathbb{A}$, since it is not equivariant. Equivariance would require that $\sigma \mathbin{\hat{\bullet}} (\pi \bullet x) = \pi \bullet (\sigma \mathbin{\hat{\bullet}} x)$, which in general only holds if σ and π commute. We do, however, have the following:

Proposition 4.5. *Let $\sigma : \mathbb{A} \to \mathbb{A}$ be a bijection. Then $\sigma \mathbin{\hat{\bullet}} (-) : \widehat{\sigma}(X) \to X$ is a natural isomorphism.*

Proof. See Appendix A. ☐

The next lemma deals with the situation where we have two permutations $\sigma_1, \sigma_2 : \mathbb{A} \to \mathbb{A}$.

Lemma 4.6. *Consider permutations $\sigma_1, \sigma_2 : \mathbb{A} \to \mathbb{A}$, and let $\sigma = \sigma_2 \circ \sigma_1$. Let $(X, \bullet)$ be an object of $\mathbf{Nom}_\mathbb{A}$, and let $\widehat{\sigma}_1(X, \bullet) = (X, \diamond)$ as in Definition 4.3. Then the following diagram commutes:*

$$\begin{tikzcd}
& \widehat{\sigma}_2(\widehat{\sigma}_1(X)) \arrow[d, equal] \arrow[ddl, "\widehat{\sigma}_2(\sigma_1 \mathbin{\hat{\bullet}} (-))"'] \arrow[ddr, "\sigma_2 \mathbin{\hat{\diamond}} (-)"] & \\
& \widehat{\sigma}(X) \arrow[dd, "\sigma \mathbin{\hat{\bullet}} (-)"] & \\
\widehat{\sigma}_2(X) \arrow[dr, "\sigma_2 \mathbin{\hat{\bullet}} (-)"'] & & \widehat{\sigma}_1(X) \arrow[dl, "\sigma_1 \mathbin{\hat{\bullet}} (-)"] \\
& X &
\end{tikzcd}$$

Proof. The morphisms along the left and right of this diagram map x to $\sigma_2 \mathbin{\hat{\bullet}} (\sigma_1 \mathbin{\hat{\bullet}} x)$ and $\sigma_1 \mathbin{\hat{\bullet}} (\sigma_2 \mathbin{\hat{\diamond}} x)$, respectively. By the definition of the various actions, both are equal to $\sigma \mathbin{\hat{\bullet}} x$. Alternatively, we could have also observed that the outer square commutes by naturality. ☐

Lemma 4.7. *Let $\mu : \mathbb{A} \to \mathbb{B}$ and $\sigma : \mathbb{A} \to \mathbb{A}$ be bijections. Let $(X, \bullet)$ be an object of* $\mathbf{Nom}_{\mathbb{A}}$*, and let $\widehat{\mu}(X, \bullet) = (X, \diamond)$ as in Definition 4.3. Then $\mu\sigma\mu^{-1} \mathbin{\hat{\diamond}} (-) = \widehat{\mu}(\sigma \mathbin{\hat{\bullet}} (-))$, i.e., the following diagram commutes in* $\mathbf{Nom}_{\mathbb{B}}$.

$$\begin{tikzcd}
& \widehat{\mu\sigma}(X) \arrow[dl, equal] \arrow[dr, equal] & \\
\widehat{\mu}(\widehat{\sigma}(X)) \arrow[dr, "\widehat{\mu}(\sigma \mathbin{\hat{\bullet}} (-))"'] & & \widehat{\mu\sigma\mu^{-1}}(\widehat{\mu}(X)) \arrow[dl, "\mu\sigma\mu^{-1} \mathbin{\hat{\diamond}} (-)"] \\
& \widehat{\mu}(X) &
\end{tikzcd}$$

Proof. Let $x \in \widehat{\mu\sigma}(X)$. Then $(\mu\sigma\mu^{-1}) \mathbin{\hat{\diamond}} x = \mu^{-1}(\mu\sigma\mu^{-1})\mu \bullet x = \sigma \bullet x$. Also, since the $\widehat{\mu}$ functor acts trivially on the underlying function, $\widehat{\mu}(\sigma \mathbin{\hat{\bullet}} (-))$ acts the same as $\sigma \mathbin{\hat{\bullet}} (-)$ on elements. ☐

In this paper, we will rarely deal with two completely arbitrary atom sets $\mathbb{A}$ and $\mathbb{B}$; instead, we will most often use the functor $\widehat{\sigma}$ in the context of subsets of a fixed atom set $\mathbb{A}$.

Notation 4.8. Fix an atom set $\mathbb{A}$. Whenever $S \subseteq \mathbb{A}$ is a finite subset, we write $\mathbb{A}_S = \mathbb{A} - S$.

Note that every permutation $\sigma : \mathbb{A} \to \mathbb{A}$ (whether or not it is finitely supported) restricts to a bijective function $\sigma : \mathbb{A}_S \xrightarrow{\cong} \mathbb{A}_{\sigma \mathbin{\hat{\bullet}} S}$. By Definition 4.3, this in turn induces an isomorphism of categories

$$\widehat{\sigma} : \mathbf{Nom}_{\mathbb{A}_S} \xrightarrow{\cong} \mathbf{Nom}_{\mathbb{A}_{\sigma \mathbin{\hat{\bullet}} S}}.$$

Remark 4.9. In this section, we considered that a bijection of atom sets $\mu : \mathbb{A} \to \mathbb{B}$ induces an isomorphism of categories $\widehat{\mu} : \mathbf{Nom}_{\mathbb{A}} \to \mathbf{Nom}_{\mathbb{B}}$. In particular, the categories $\mathbf{Nom}_{\mathbb{A}}$ and $\mathbf{Nom}_{\mathbb{B}}$ for two different countable atom sets are always equivalent. It is an interesting observation that even if we were to allow atom sets of arbitrary infinite cardinalities, the categories $\mathbf{Nom}_{\mathbb{A}}$ and $\mathbf{Nom}_{\mathbb{B}}$ still end up being equivalent. We do not need this more general fact in this paper, and we will not prove it here. However, it justifies that working with a countable atom set is without loss of generality.

5 Nominal families

Fix a set $\mathbb{A}$ of atoms, and let X be an object in $\mathbf{Nom}_{\mathbb{A}}$. Consider the slice category $\mathbf{Nom}_{\mathbb{A}}/X$ and an object in it:

$$\begin{tikzcd}
Y \arrow[d, "p"] \\
X
\end{tikzcd}$$

Let $x \in X$ and let Y_x denote the fiber over x, i.e., $Y_x = \{y \in Y \mid p(y) = x\}$. Note that Y_x is not in general an object of $\mathbf{Nom}_{\mathbb{A}}$, because the π-action is not well-defined on it: For $y \in Y_x$, we have $\pi \bullet y \in Y_{\pi \bullet x}$ instead of $\pi \bullet y \in Y_x$.

However, Y_x is an object of $\mathbf{Nom}_{\mathbb{A}_x}$, where $\mathbb{A}_x = \mathbb{A} - \operatorname{supp}(x)$. Specifically, let $\iota : \mathbb{A}_x \hookrightarrow \mathbb{A}$ be the subset inclusion, and recall from Definition 4.1 that every permutation $\pi \in \Pi_{\mathbb{A}_x}$ can be canonically extended to a permutation $\Pi_\iota(\pi) \in \Pi_{\mathbb{A}}$. Then the action of $\pi \in \Pi_{\mathbb{A}_x}$ on an element $y \in Y_x$ is defined as

$$\pi \bullet_{Y_x} y = \Pi_\iota(\pi) \bullet_Y y.$$

We call $\mathbf{Nom}_{\mathbb{A}_x}$ the category of *nominal sets relative to* x.

Conversely, consider a family $(Y_x)_{x\in X}$ of nominal sets, where each Y_x is an object of $\mathbf{Nom}_{\mathbb{A}_x}$. With what additional structure must we equip this family, so that it will give rise to an object $p : Y \to X$ of the slice category? Clearly, we want $Y = \sum_{x\in X} Y_x$, where $\sum$ denotes the disjoint union. The additional structure we need on the family $(Y_x)_{x\in X}$ is a "global" action $\tilde{\bullet}$ such that for all $\pi \in \Pi_{\mathbb{A}}$, we have $\pi\,\tilde{\bullet}\,(-) : Y_x \to Y_{\pi\bullet x}$, subject to some axioms. This motivates the following definition.

Definition 5.1. Let X be a nominal set over the atom set $\mathbb{A}$. A *nominal family over* X is a family $(Y_x)_{x\in X}$ such that each Y_x is a nominal set in $\mathbf{Nom}_{\mathbb{A}_x}$, together with a family of operations $\pi\,\tilde{\bullet}\,(-) : Y_x \to Y_{\pi\bullet x}$, satisfying the following properties:

(a) $\pi_1\pi_2\,\tilde{\bullet}\,y = \pi_1\,\tilde{\bullet}\,(\pi_2\,\tilde{\bullet}\,y)$.

(b) If $\pi\#x$, then for all $y \in Y_x$, we have $\pi\,\tilde{\bullet}\,y = \pi \bullet y$.

Property (a) is like the usual property of an action, and property (b) states that the "global" action $\tilde{\bullet}$ coincides with the "local" action $\bullet$ on the nominal set Y_x when both are defined. Note that we did not require $\mathrm{id}\,\tilde{\bullet}\,y = y$, because this follows from (b) by setting $\pi = \mathrm{id}$.

Given two nominal families $(Y_x)_{x\in X}$ and $(Z_x)_{x\in X}$, consider a family of morphisms $f_x : Y_x \to Z_x$ where each f_x is a morphism in $\mathbf{Nom}_{\mathbb{A}_x}$. We say that such a family of morphisms is *equivariant* if it respects the global action, i.e., if for all $x \in X$, $y \in Y_x$, and $\pi \in \Pi_{\mathbb{A}}$, we have

$$\pi\,\tilde{\bullet}\,(f_x(y)) = f_{\pi\bullet x}(\pi\,\tilde{\bullet}\,y).$$

The *category of nominal families over* X has nominal families as the objects and equivalent families of morphisms as the maps. We denote it by $\mathbf{Nom}^X$. The following proposition is easy to prove.

Proposition 5.2. *Let* $X \in \mathbf{Nom}_{\mathbb{A}}$. *The categories* $\mathbf{Nom}^X$ *and* $\mathbf{Nom}/X$ *are equivalent.*

Proof. We already saw that an object $p : Y \to X$ of the slice category gives rise to a nominal family $(Y_x)_{x\in X}$. Conversely, if $(Y_x)_{x\in X}$ is a nominal family, let $Y = \sum_{x\in X} Y_x = \{(x,y) \mid x \in X, y \in Y_x\}$ be the disjoint union with $p : Y \to X$ defined by $p(x,y) = x$. We can equip Y with the action $\pi\bullet(x,y) = (\pi\bullet x, \pi\tilde{\bullet}y)$. Every element (x,y) is finitely supported by $\mathrm{supp}_{\mathbb{A}}(x) \cup \mathrm{supp}_{\mathbb{A}_x}(y)$. It is routine to verify that it is a well-defined object of $\mathbf{Nom}/X$, and that the two constructions respect morphisms and are mutually inverse up to isomorphism. □

Consider a nominal family $(Y_x)_{x\in X}$, some permutation $\pi \in \Pi_{\mathbb{A}}$ and some $x \in X$. We may ask what is the relationship between the fibers Y_x and $Y_{\pi\bullet x}$. The global action gives us functions $\pi\,\tilde{\bullet}\,(-) : Y_x \to Y_{\pi\bullet x}$ and $\pi^{-1}\,\tilde{\bullet}\,(-) : Y_{\pi\bullet x} \to Y_x$. Since these functions are each other's inverses, it follows that Y_x and $Y_{\pi\bullet x}$ are isomorphic as sets. Are they isomorphic as nominal sets? The question does not really make sense, because Y_x and $Y_{\pi\bullet x}$ are objects in two different categories, namely in $\mathbf{Nom}_{\mathbb{A}_x}$ and $\mathbf{Nom}_{\mathbb{A}_{\pi\bullet x}}$, respectively. On the other hand, as we saw in Section 4.2, there is an isomorphism of categories $\widehat{\pi} : \mathbf{Nom}_{\mathbb{A}_x} \to \mathbf{Nom}_{\mathbb{A}_{\pi\bullet x}}$. Modulo this isomorphism, the objects Y_x and $Y_{\pi\bullet x}$ are isomorphic:

Proposition 5.3. *Given a nominal family* $(Y_x)_{x\in X}$, *some* $\pi \in \Pi_{\mathbb{A}}$ *and* $x \in X$. *Then the function* $\varphi_{\pi,x} : \widehat{\pi}(Y_x) \to Y_{\pi\bullet x}$ *defined by* $\varphi_{\pi,x}(y) = \pi\,\tilde{\bullet}\,y$ *is an equivariant isomorphism.*

Proof. Clearly $\varphi_{\pi,x}$ is invertible, with its inverse given by the action of π^{-1}. We must show that $\varphi_{\pi,x}$ is equivariant. Recall from Section 4.2 that $\widehat{\pi}(Y_x) = (Y_x, \diamond)$, with the action $\rho \diamond y = (\pi^{-1}\rho\pi) \bullet y$. Consider some $y \in Y_x$ and $\rho \in \Pi_{\pi\bullet x}$. We have

$$\varphi_{\pi,x}(\rho \diamond y) = \pi\,\tilde{\bullet}\,((\pi^{-1}\rho\pi) \bullet y) = \pi\,\tilde{\bullet}\,(\pi^{-1}\rho\pi)\,\tilde{\bullet}\,y = \pi\,\tilde{\bullet}\,\pi^{-1}\,\tilde{\bullet}\,\rho\,\tilde{\bullet}\,\pi\,\tilde{\bullet}\,y = \rho\,\tilde{\bullet}\,\pi\,\tilde{\bullet}\,y = \rho \bullet (\pi\,\tilde{\bullet}\,y) = \rho \bullet \varphi_{\pi,x}(y).$$

Here, in the relevant steps, we were able to replace $\bullet$ by $\tilde{\bullet}$ and vice versa because $(\pi^{-1}\rho\pi)\#x$ and $\rho\#(\pi \bullet x)$. □

Remark 5.4. We can equivalently define a nominal family in terms of a family of maps $\varphi_{\pi,x}$, rather than a global action $\tilde{\bullet}$. In this case, a nominal family over X consists of a family $(Y_x)_{x\in X}$, where $Y_x \in \mathbf{Nom}_{\mathbb{A}_x}$, together with an equivariant map $\varphi_{\pi,x} : \widehat{\pi}(Y_x) \to Y_{\pi\bullet x}$ for each $\pi \in \Pi_{\mathbb{A}}$ and $x \in X$, satisfying the following conditions:

(a) $\varphi_{\pi_1\pi_2,x} = \varphi_{\pi_1,\pi_2\bullet x}\circ\widehat{\pi_1}(\varphi_{\pi_2,x})$.

(b) If $\pi\#x$, then $\varphi_{\pi,x}(y) = \pi\bullet y$ for all $y\in\widehat{\pi}(Y_x)$.

The equivalence between this definition and Definition 5.1 is straightforward, as both sets of conditions refer to the same sets and elements. Also, due to Proposition 5.3, conditions (a) and (b) automatically imply that the maps $\varphi_{\pi,x}$ are equivariant and invertible.

In this case, a family of morphisms $f_x : Y_x \to Z_x$ is equivariant if and only if the following diagram commutes for all $x\in X$ and $\pi\in\Pi_{\mathbb{A}}$:

$$\begin{tikzcd}
\widehat{\pi}(Y_x) \arrow[r, "\widehat{\pi}(f_x)"] \arrow[d, "\varphi^Y_{\pi,x}"'] & \widehat{\pi}(Z_x) \arrow[d, "\varphi^Z_{\pi,x}"] \\
Y_{\pi\bullet x} \arrow[r, "f_{\pi\bullet x}"] & Z_{\pi\bullet x}
\end{tikzcd}$$

6 Monoidal structure

In this section, we will discuss the relation between the monoidal structure on **Nom** and the monoidal structure on **Nom**$/X$.

Theorem 6.1. *Every monoidal structure $\otimes$ on* **Nom** *induces a monoidal structure $\otimes_X$ on* **Nom**$/X$*, which is uniquely determined up to coherent isomorphism.*

Before we prove the theorem, we need to review some facts about how monoidal structures transport along isomorphisms.

6.1 Monoidal structures and isomorphisms of categories

Recall that a functor $F : \mathbf{C}\to\mathbf{D}$ between monoidal categories is said to be a *(strong) monoidal functor* if it is equipped with natural isomorphisms $m^0 : FI \to I$ and $m^{A,B} : F(A\otimes B)\to FA\otimes FB$ that are compatible with the associators $\alpha_{A,B,C}$ and left and right unitors λ_A and ρ_A in the obvious way. Note that in the literature, it is more common to define the arrows m^0 and $m^{A,B}$ to point in the opposite direction, but since they are isomorphisms, this does not matter. It is more convenient for our purposes to define them as above. Also, since we are never interested in lax (nor op-lax) monoidal functors, we will drop the adjective “strong”.

Consider a category $\mathbf{C}$ that is equipped with two monoidal structures $(\otimes, I, \lambda, \rho, \alpha)$ and $(\otimes', I', \lambda', \rho', \alpha')$. In this case, an *isomorphism* between the monoidal structures is a monoidal functor $F : (\mathbf{C}, \otimes, I, \lambda, \rho, \alpha) \to (\mathbf{C}, \otimes', I', \lambda', \rho', \alpha')$ whose underlying functor is the identity. Specifically, this means that there are natural isomorphisms $m^0 : I \to I'$ and $m^{A,B} : A\otimes B\to A\otimes' B$ that are compatible with the associators and unitors.

We now consider how monoidal structures are transported along isomorphisms of categories. Most of what we say below could be generalized to equivalences of categories, but for the purposes of this paper, it is sufficient to consider isomorphisms.

Let $F : \mathbf{C}\to\mathbf{D}$ be an isomorphism of categories. Then every monoidal structure $(\otimes, I, \lambda, \rho, \alpha)$ on $\mathbf{C}$ canonically induces a monoidal structure $(\otimes_F, I_F, \lambda_F, \rho_F, \alpha_F)$ on $\mathbf{D}$ in the obvious way. For example, $A\otimes_F B = F(F^{-1}(A)\otimes F^{-1}(B))$, $I_F = FI$, etc.

In general, if $F, G : \mathbf{C}\to\mathbf{D}$ are two isomorphisms of categories, then the two induced monoidal structures $\otimes_F$ and $\otimes_G$ may not be isomorphic. For example, consider $\mathbf{C} = \mathbf{D} = \mathbf{Set}\times\mathbf{Set}$ with the monoidal structure $(A, B)\otimes(A', B') = (A\times A', B + B')$. Then the functors $F(A, B) = (A, B)$ and $G(A, B) = (B, A)$ induce two non-isomorphic monoidal structures on $\mathbf{Set}\times\mathbf{Set}$.

If, on the other hand, the functors F, G are naturally isomorphic to each other, say via $\eta^A : FA \to GA$, then the monoidal structures induced on $\mathbf{D}$ by F and G are isomorphic in a canonical way.

6.2 Transport of monoidal structure from $\mathbf{Nom}_{\mathbb{A}}$ *to* $\mathbf{Nom}_{\mathbb{B}}$

We now want to show that if $\mathbb{A}$ and $\mathbb{B}$ are two (countably infinite) atom sets, then any monoidal structure on $\mathbf{Nom}_{\mathbb{A}}$ determines a monoidal structure on $\mathbf{Nom}_{\mathbb{B}}$, and moreover, the latter structure is uniquely determined up to isomorphisms of monoidal structures (and in particular, it does not depend on any particular choice of a bijection between $\mathbb{A}$ and $\mathbb{B}$).

So consider a monoidal structure on $\mathbf{Nom}_{\mathbb{A}}$. Clearly, any bijection $\gamma : \mathbb{A} \to \mathbb{B}$ induces an isomorphism between $\mathbf{Nom}_{\mathbb{A}}$ and $\mathbf{Nom}_{\mathbb{B}}$, and therefore induces a monoidal structure on $\mathbf{Nom}_{\mathbb{B}}$. What we must show is that the latter monoidal structure is independent of γ.

Proposition 6.2. *Let $\gamma_1, \gamma_2 : \mathbb{A} \to \mathbb{B}$ be two bijections and let $\otimes_1$ and $\otimes_2$ be the monoidal structures on $\mathbf{Nom}_{\mathbb{B}}$ induced by γ_1 and γ_2, respectively. The monoidal structures $\otimes_1$ and $\otimes_2$ are isomorphic.*

Proof. Define $\sigma : \mathbb{B} \to \mathbb{B}$ by $\sigma = \gamma_2 \circ (\gamma_1)^{-1}$. Consider $\widehat{\sigma} : \mathbf{Nom}_{\mathbb{B}} \to \mathbf{Nom}_{\mathbb{B}}$ as in Section 4.2. Note that $\widehat{\sigma}$ is naturally isomorphic to the identity functor $\mathrm{id} : \mathbf{Nom}_{\mathbb{B}} \to \mathbf{Nom}_{\mathbb{B}}$ by Proposition 4.5. Therefore, by whiskering, $\widehat{\sigma} \circ \widehat{\gamma}_1 = \widehat{\gamma}_2$ is naturally isomorphic to $\mathrm{id} \circ \widehat{\gamma}_1 = \widehat{\gamma}_1$. As noted in Section 6.1, this implies that the monoidal structures on $\mathbf{Nom}_{\mathbb{B}}$ induced by $\widehat{\gamma}_2$ and $\widehat{\gamma}_1$ are isomorphic, as claimed. ☐

Remark 6.3. We write $\eta^{A,B}_{\gamma_1,\gamma_2} : A \otimes_1 B \to A \otimes_2 B$ for the natural isomorphism described in Proposition 6.2. It satisfies the following diagram, where $\sigma = \gamma_2 \circ (\gamma_1)^{-1}$:

$$\begin{tikzcd}
\widehat{\sigma}(A \otimes_1 B) \arrow[r, equal] \arrow[d, "\sigma \mathbin{\hat{\bullet}} (-)"', "\cong"] & \widehat{\sigma}(A) \otimes_2 \widehat{\sigma}(B) \arrow[d, "\cong"', "\sigma \mathbin{\hat{\bullet}} (-) \otimes_2 \sigma \mathbin{\hat{\bullet}} (-)"] \\
A \otimes_1 B \arrow[r, "\eta^{A,B}_{\gamma_1,\gamma_2}"] & A \otimes_2 B.
\end{tikzcd}$$

Similarly, we write $\eta^{0}_{\gamma_1,\gamma_2} : I_1 \to I_2$ for the corresponding isomorphism between the units of the two monoidal structures. It satisfies the following:

$$\begin{tikzcd}
\widehat{\sigma}(I_1) \arrow[r, equal] \arrow[d, "\sigma \mathbin{\hat{\bullet}} (-)"', "\cong"] & I_2 \arrow[d, equal] \\
I_1 \arrow[r, "\eta^{0}_{\gamma_1,\gamma_2}"] & I_2.
\end{tikzcd}$$

Proposition 6.2 ensures that the monoidal structures induced on $\mathbf{Nom}_{\mathbb{B}}$ by various different bijections $\mathbb{A} \to \mathbb{B}$ are all isomorphic to each other. We will also need these isomorphisms to be coherent, i.e., any three such monoidal structures are isomorphic in a unique way. The following lemma shows this.

Lemma 6.4. *Let $\gamma_1, \gamma_2, \gamma_3 : \mathbb{A} \to \mathbb{B}$ be three bijections and let $\otimes_1$, $\otimes_2$ and $\otimes_3$ be the monoidal structures on $\mathbf{Nom}_{\mathbb{B}}$ induced by γ_1, γ_2, and γ_3, respectively. Then $\eta^{A,B}_{\gamma_1,\gamma_3} = \eta^{A,B}_{\gamma_2,\gamma_3} \circ \eta^{A,B}_{\gamma_1,\gamma_2}$.*

Proof. See Appendix A. ☐

Lemma 6.5. *Let $\gamma_1, \gamma_2 : \mathbb{A} \to \mathbb{B}$ be bijections, inducing monoidal structures $\otimes_1$ and $\otimes_2$ on $\mathbf{Nom}_{\mathbb{B}}$. Let $\pi : \mathbb{B} \to \mathbb{C}$ be another bijection, and let $\otimes_1'$, and $\otimes_2'$ be the monoidal structures induced on $\mathbf{Nom}_{\mathbb{C}}$ by $\pi\gamma_1$, and $\pi\gamma_2$, respectively. Then the map $\eta^{A,B}_{\gamma_1,\gamma_2}$ is compatible with the $\widehat{\pi}$ functor, i.e., the following diagram commutes.*

$$\begin{tikzcd}
\widehat{\pi}(A \otimes_1 B) \arrow[r, "\widehat{\pi}(\eta^{A,B}_{\gamma_1,\gamma_2})"'] \arrow[d, equal] & \widehat{\pi}(A \otimes_2 B) \arrow[d, equal] \\
\widehat{\pi}(A) \otimes_1' \widehat{\pi}(B) \arrow[r, "\eta^{A,B}_{\pi\gamma_1,\pi\gamma_2}"'] & \widehat{\pi}(A) \otimes_2' \widehat{\pi}(B)
\end{tikzcd}$$

Proof. See Appendix A. ☐

6.3 $\widehat{\mu}$ is a monoidal functor

As usual, let $\otimes$ be a monoidal structure on $\mathbf{Nom}_{\mathbb{A}}$. Consider bijections $\delta_1 : \mathbb{A} \to \mathbb{B}$ and $\delta_2 : \mathbb{A} \to \mathbb{C}$, and let $\otimes_1$ and $\otimes_2$ be the monoidal structures induced by δ_1 and δ_2 on $\mathbf{Nom}_{\mathbb{B}}$ and $\mathbf{Nom}_{\mathbb{C}}$, respectively.

Now consider another arbitrary bijection $\mu : \mathbb{B} \to \mathbb{C}$, inducing an isomorphism of categories $\widehat{\mu} : \mathbf{Nom}_{\mathbb{B}} \to \mathbf{Nom}_{\mathbb{C}}$. Note that we do not require $\delta_2 = \mu\delta_1$. Therefore, by transporting $\otimes_1$ along $\widehat{\mu}$, we obtain another monoidal structure on $\mathbf{Nom}_{\mathbb{C}}$, which we call $\otimes_2'$, and which is not necessarily the same as $\otimes_2$. By definition, $\widehat{\mu}$ strictly maps $\otimes_1$ to $\otimes_2'$, and therefore it certainly is a monoidal functor with respect to $\otimes_1$ and $\otimes_2'$. Less obvious is the fact that $\widehat{\mu}$ is also canonically a monoidal functor with respect to $\otimes_1$ and $\otimes_2$. The purpose of this subsection is to spell out how.

The key is Proposition 6.2, which shows that the two monoidal structures on $\mathbf{Nom}_{\mathbb{C}}$, $\otimes_2$ and $\otimes_2'$, while not the same, are canonically isomorphic. Therefore, $\widehat{\mu}$ can be canonically equipped with the structure of a monoidal functor with respect to $\otimes_1$ and $\otimes_2$. Explicitly, we define a family of maps $m_\mu : \widehat{\mu}(A \otimes_1 B) \to \widehat{\mu}(A) \otimes_2 \widehat{\mu}(B)$ by the following diagram:

$$\begin{tikzcd}
\widehat{\mu}(A \otimes_1 B) \arrow[r, equal] \arrow[rr, bend right, "m_\mu"'] & \widehat{\mu}(A) \otimes_2' \widehat{\mu}(B) \arrow[r, "\eta_{\mu\circ\delta_1,\delta_2}"] & \widehat{\mu}(A) \otimes_2 \widehat{\mu}(B).
\end{tikzcd}$$

Similarly, we also define

$$\begin{tikzcd}
\widehat{\mu}(I_1) \arrow[r, equal] \arrow[rr, bend right, "m_\mu^0"'] & I_2' \arrow[r, "\eta^0_{\mu\circ\delta_1,\delta_2}"] & I_2.
\end{tikzcd}$$

We note that m_μ is a natural isomorphism because the equality $\widehat{\mu}(A \otimes_1 B) = \widehat{\mu}(A) \otimes_2' \widehat{\mu}(B)$ is natural by definition of $\otimes_2'$, and $\eta_{\mu\circ\delta_1,\delta_2}$ is natural by Remark 6.3 and the proof of Proposition 6.2. Also, m_μ is compatible with the monoidal associators and unitors. Indeed, the compatibility of the equality $\widehat{\mu}(A \otimes_1 B) = \widehat{\mu}(A) \otimes_2' \widehat{\mu}(B)$ holds by the definition of the associators and unitors for $\otimes_2'$, whereas the compatibility for $\eta_{\mu\circ\delta_1,\delta_2}$ holds because it is an isomorphism of monoidal structures by Proposition 6.2. Thus, m_μ indeed makes $\widehat{\mu}$ into a monoidal functor from $(\mathbf{Nom}_{\mathbb{B}}, \otimes_1)$ to $(\mathbf{Nom}_{\mathbb{C}}, \otimes_2)$.

The following lemma establishes a useful property of the m maps.

Lemma 6.6. *Let $\delta_1 : \mathbb{A} \to \mathbb{B}$, $\delta_2 : \mathbb{A} \to \mathbb{C}$, and $\delta_3 : \mathbb{A} \to \mathbb{D}$ be three bijections, inducing monoidal structures $\otimes_1$, $\otimes_2$, and $\otimes_3$ on $\mathbf{Nom}_{\mathbb{B}}$, $\mathbf{Nom}_{\mathbb{C}}$, and $\mathbf{Nom}_{\mathbb{D}}$, respectively. Let $\mu_1 : \mathbb{B} \to \mathbb{C}$ and $\mu_2 : \mathbb{C} \to \mathbb{D}$ be bijections. Then the following diagram commutes.*

$$\begin{tikzcd}
\widehat{\mu}_1\widehat{\mu}_2(A \otimes_1 B) \arrow[r, "\widehat{\mu}_1(m_{\mu_2})"] \arrow[d, equal] & \widehat{\mu}_1(\widehat{\mu}_2(A) \otimes_2 \widehat{\mu}_2(B)) \arrow[r, "m_{\mu_1}"] & \widehat{\mu}_1\widehat{\mu}_2(A) \otimes_3 \widehat{\mu}_1\widehat{\mu}_2(B) \arrow[d, equal] \\
\widehat{\mu_1\mu_2}(A \otimes_1 B) \arrow[rr, "m_{\mu_1\mu_2}"'] & & \widehat{\mu_1\mu_2}(A) \otimes_3 \widehat{\mu_1\mu_2}(B)
\end{tikzcd}$$

Proof. See Appendix A. ☐

A special case arises when $\delta_1 = \delta_2$. In this case, we have the following:

Lemma 6.7. *Let $\delta_1 = \delta_2 : \mathbb{A} \to \mathbb{B}$, so that the induced tensors $\otimes_1$ and $\otimes_2$ on $\mathbf{Nom}_{\mathbb{A}}$ coincide. Consider any $\mu : \mathbb{B} \to \mathbb{B}$. Then we have*

$$\begin{tikzcd}
\widehat{\mu}(A \otimes_1 B) \arrow[r, "m_\mu"] \arrow[d, "\mu \mathbin{\hat{\bullet}} (-)"'] & \widehat{\mu}(A) \otimes_2 \widehat{\mu}(B) \arrow[d, "\mu \mathbin{\hat{\bullet}} (-) \otimes_2 \mu \mathbin{\hat{\bullet}} (-)"] \\
A \otimes_1 B \arrow[r, equal] & A \otimes_2 B.
\end{tikzcd}$$

Proof. See Appendix A. □

6.4 *Proof of Theorem 6.1*

Consider a monoidal structure $\otimes$ on $\mathbf{Nom}$. We wish to define a monoidal structure $\otimes_X$ on $\mathbf{Nom}/X$. From Proposition 5.2, we know that $\mathbf{Nom}/X$ is equivalent to the category $\mathbf{Nom}^X$ of nominal families over X. It therefore suffices to define a monoidal structure on the latter.

6.4.1 *Definition of $A \otimes_X B$*

Consider two nominal families $A = (A_x)_{x\in X}$ and $B = (B_x)_{x\in X}$. We write $\pi \mathbin{\tilde{\bullet}} (-)$ for the "global" action on both of these families. We wish to define a nominal family $A \otimes_X B = (C_x)_{x\in X}$.

We start by choosing a fixed but arbitrary bijection $\delta_x : \mathbb{A} \to \mathbb{A}_x$ for every $x \in X$. This induces an isomorphism of categories $\widehat{\delta_x} : \mathbf{Nom}_{\mathbb{A}} \to \mathbf{Nom}_{\mathbb{A}_x}$, and therefore a monoidal structure in $\mathbf{Nom}_{\mathbb{A}_x}$. We write $\otimes_x$ for this monoidal structure (and also I_x, α_x, etc., as necessary).

Next, we define $C_x = A_x \otimes_x B_x$. To make this into a nominal family, we must define a global π-action, i.e., a family of operations $\pi \mathbin{\tilde{\bullet}} (-) : C_x \to C_{\pi\bullet x}$. So fix some permutation π and an $x \in X$. For ease of notation, let us write $y = \pi \bullet x$.

Each of the sets A_x and B_x is already equipped with an action, namely $\pi \mathbin{\tilde{\bullet}} (-) : A_x \to A_y$ and $\pi \mathbin{\tilde{\bullet}} (-) : B_x \to B_y$. By Proposition 5.3, these correspond to equivariant maps $\varphi^A_{\pi,x} : \widehat{\pi}(A_x) \to A_y$ and $\varphi^B_{\pi,x} : \widehat{\pi}(B_x) \to B_y$. Since both of these maps are morphisms in the category $\mathbf{Nom}_{\mathbb{A}_y}$, we may tensor them to get a map

$$\varphi^A_{\pi,x} \otimes_y \varphi^B_{\pi,x} \;:\; \widehat{\pi}(A_x) \otimes_y \widehat{\pi}(B_x) \;\to\; A_y \otimes_y B_y.$$

We now define the map $\varphi^C_{\pi,x} : \widehat{\pi}(C_x) \to C_y$ as follows, using the m-map of Section 6.3.

$$\begin{tikzcd}
\widehat{\pi}(C_x) \arrow[rr, "\varphi^C_{\pi,x}"] \arrow[d, equal] & & C_y \arrow[d, equal] \\
\widehat{\pi}(A_x \otimes_x B_x) \arrow[r, "m_\pi"] & \widehat{\pi}(A_x) \otimes_y \widehat{\pi}(B_x) \arrow[r, "\varphi^A_{\pi,x} \otimes_y \varphi^B_{\pi,x}"] & A_y \otimes_y B_y
\end{tikzcd}$$

By Remark 5.4, the family of maps $\varphi^C_{\pi,x}$ is sufficient to turn $(C_x)_{x\in X}$ into a nominal family, provided that it satisfies the remark's properties (a) and (b). This is shown in Appendix B. This finishes the proof that $A \otimes_X B$ is a well-defined object.

6.4.2 *Definition of $f \otimes_X g$ and functoriality*

To continue our definition of the monoidal structure on $\mathbf{Nom}/X$, we must now define the morphism part of the monoidal structure. Therefore, consider $f : A \to A'$ and $g : B \to B'$ in $\mathbf{Nom}/X$. We will define a morphism $f \otimes_X g : A \otimes_X B \to A' \otimes_X B'$ as a family of maps $(f \otimes_X g)_x : (A \otimes_X B)_x \to (A' \otimes_X B')_x$. They are defined by

$$\begin{tikzcd}
(A \otimes_X B)_x \arrow[r, "(f \otimes_X g)_x"] \arrow[d, equal] & (A' \otimes_X B')_x \arrow[d, equal] \\
A_x \otimes_x B_x \arrow[r, "f_x \otimes_x g_x"] & A'_x \otimes_x B'_x.
\end{tikzcd}$$

To make sure that this is a well-defined morphism of nominal families, by Remark 5.4, it is sufficient to show that these maps are compatible with the family of maps $\varphi^{A\otimes_X B}_{\pi,x}$, which we henceforth refer to as the φ-maps. In other words, we must show that the outer part of the following diagram commutes for all

$\pi \in \Pi_{\mathbb{A}}$, where $y = \pi \bullet x$.

$$\begin{tikzcd}
\widehat{\pi}(A_x \otimes_x B_x) \arrow[rrr, "\widehat{\pi}(f_x \otimes_x g_x)"] \arrow[rrr, phantom, "(1)" swap] \arrow[dr, "m_\pi"] \arrow[dd, "\varphi^{A\otimes_X B}_{\pi,x}" swap] \arrow[dd, phantom, "(2)"] & & & \widehat{\pi}(A'_x \otimes_x B'_x) \arrow[dl, "m_\pi" swap] \arrow[dd, "\varphi^{A'\otimes_X B'}_{\pi,x}"] \arrow[dd, phantom, "(2)" swap] \\
& \widehat{\pi}(A_x) \otimes_y \widehat{\pi}(B_x) \arrow[r, "\widehat{\pi}(f_x) \otimes_y \widehat{\pi}(g_x)"] \arrow[r, phantom, "(3)" swap] \arrow[dl, "\varphi^{A}_{\pi,x} \otimes_y \varphi^{B}_{\pi,x}"] & \widehat{\pi}(A'_x) \otimes_y \widehat{\pi}(B'_x) \arrow[dr, "\varphi^{A'}_{\pi,x} \otimes_y \varphi^{B'}_{\pi,x}" swap] & \\
A_y \otimes_y B_y \arrow[rrr, "f_y \otimes_y g_y"] & & & A'_y \otimes_y B'_y.
\end{tikzcd}$$

Part (1) commutes because m_π is a natural transformation; the parts labelled (2) commute by definition of the φ-maps for $\otimes_X$; part (3) commutes since f and g are morphisms of nominal families, and are therefore compatible with their respective φ-maps. This finishes the proof that $f \otimes_X g : A \otimes_X B \to A' \otimes_X B'$ is a well-defined morphism.

Next, we must show that $\otimes_X$ is a functor. However, this is straightforward because the required equations hold fiberwise: we have $\mathrm{id}^A_x \otimes_x \mathrm{id}^B_x = \mathrm{id}^{A\otimes B}_x$ and $(f_x \otimes_x g_x) \circ (f'_x \otimes_x g'_x) = (f_x \circ f'_x) \otimes_x (g_x \circ g'_x)$, because $\otimes_x$ is a functor on $\mathbf{Nom}_{\mathbb{A}_x}$ for every $x \in X$.

6.4.3 Definition of monoidal unit, associators, and unitors

To continue the definition of the monoidal structure on $\mathbf{Nom}/X$, we must provide a monoidal unit I, which we simply take to be the nominal family $(I_x)_{x\in X}$, where I_x is the monoidal unit of $\otimes_x$ in $\mathbf{Nom}_{\mathbb{A}_x}$. This becomes a nominal family via $\varphi_{\pi,x} = m^0_\pi : \widehat{\pi}(I_x) \to I_{\pi\bullet x}$.

We must also provide associators $\alpha^{A,B,C} : (A \otimes_X B) \otimes_X C \to A \otimes_X (B \otimes_X C)$ and unitors $\lambda^A : I \otimes_X A \to A$ and $\rho^A : A \otimes_X I \to A$. We define these maps fiberwise, by taking each component α_x, λ_x, and ρ_x to be an associator or unitor of the corresponding monoidal structure $\otimes_x$ on $\mathbf{Nom}_{\mathbb{A}_x}$:

$$\begin{aligned}
\alpha^{A,B,C}_x &: (A_x \otimes_x B_x) \otimes_x C_x \to A_x \otimes_x (B_x \otimes_x C_x), \\
\lambda^A_x &: I_x \otimes_x A_x \to A_x, \\
\rho^A_x &: A_x \otimes_x I_x \to A_x.
\end{aligned}$$

We must show that these maps are indeed morphisms of nominal families. By Remark 5.4, it suffices to show that they are compatible with the respective φ-maps. This is shown in Appendix C. Then $\alpha^{A,B,C}$, λ^A, and ρ^A are well-defined morphisms of $\mathbf{Nom}/X$.

6.4.4 Naturality and coherence

To finish our definition of the monoidal structure on $\mathbf{Nom}/X$, we must show that α, λ, and ρ are natural transformations, and that they satisfy the required coherence diagrams for monoidal categories, i.e., one pentagon and one triangle. However, all of these properties are trivial because they hold separately in each fiber.

This finishes the proof of Theorem 6.1.

7 Nom is locally monoidal closed

Our goal is to show that the category **Nom** is locally monoidal closed. More specifically, our main theorem states that every monoidal closed structure on **Nom** induces a monoidal closed structures on $\mathbf{Nom}/X$.

Theorem 7.1. *Every monoidal closed structure on* **Nom** *induces a monoidal closed structure on* $\mathbf{Nom}/X$.

Before we prove this theorem, we first recall some basic facts about monoidal functors between monoidal closed categories.

7.1 Monoidal functors between monoidal closed categories

Let $(\mathbf{C}, \otimes, \multimap)$ and $(\mathbf{D}, \otimes', \multimap')$ be monoidal closed categories. It is well-known that a monoidal functor $F : \mathbf{C} \to \mathbf{D}$ induces a natural transformation $n^{A,B} : F(A \multimap B) \to FA \multimap' FB$. Specifically, $n^{A,B}$ is the unique map making the following diagram commute:

$$\begin{tikzcd}
F(A \multimap B) \otimes' F(A) \arrow[r, "(m^{A \multimap B, A})^{-1}"] \arrow[d, "n^{A,B} \otimes' \mathrm{id}"'] & F((A \multimap B) \otimes A) \arrow[d, "F(\varepsilon)"] \\
(F(A) \multimap' F(B)) \otimes' F(A) \arrow[r, "\varepsilon'"'] & F(B)
\end{tikzcd}$$

Here, m is the natural isomorphism making F into a monoidal functor, and ε and ε' are the application morphisms for the respective monoidal closed structures. Equivalently, $n^{A,B} = (F(\varepsilon) \circ m_{A,B})^c$, where $(-)^c : (A \otimes' B, C) \xrightarrow{\cong} (A, B \multimap' C)$ is the currying operation. Moreover, if F is isomorphism of categories, then $n^{A,B}$ is an isomorphism.

Remark 7.2. If $(\mathbf{C}, \otimes, \multimap)$, $(\mathbf{D}, \otimes', \multimap')$, and $(\mathbf{E}, \otimes'', \multimap'')$ are monoidal closed categories and $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ are monoidal functors, the following diagram commutes.

$$\begin{tikzcd}
GF(B \multimap C) \arrow[rr, "n_{GF}"] \arrow[dr, "G(n_F)"'] & & GFB \multimap'' GFC \\
& G(FB \multimap' FC) \arrow[ur, "n_G"'] &
\end{tikzcd}$$

We also have $n_{\mathrm{id}} = \mathrm{id}$, where $\mathrm{id} : \mathbf{C} \to \mathbf{C}$ is the identity monoidal functor.

7.2 Proof of Theorem 7.1

Fix an atom set $\mathbb{A}$ and let $\mathbf{Nom} = \mathbf{Nom}_{\mathbb{A}}$ be the category of nominal sets over $\mathbb{A}$. Let X be an object of $\mathbf{Nom}$. Consider any monoidal closed structure $(\otimes, \multimap)$ on $\mathbf{Nom}$. We wish to define a monoidal closed structure $(\otimes_X, \multimap_X)$ on $\mathbf{Nom}/X$. Since $\mathbf{Nom}/X$ is equivalent to the category $\mathbf{Nom}^X$ of nominal families over X, it suffices to define a monoidal closed structure on the latter.

The tensor $\otimes_X$ was already defined in Section 6; recall that its definition required some fixed, but arbitrary choices of bijections $\delta_x : \mathbb{A} \to \mathbb{A}_x$, but the resulting tensor was shown to be well-defined up to isomorphism.

We now define the closed structure in several steps.

7.2.1 Definition of $A \multimap_X B$

Consider any two objects $A, B \in \mathbf{Nom}^X$. We must define an object $A \multimap_X B$ of $\mathbf{Nom}^X$. By definition, A and B are nominal families, say $A = (A_x, \varphi^A_{\pi,x})$ and $B = (B_x, \varphi^B_{\pi,x})$, where $A_x, B_x \in \mathbf{Nom}_{\mathbb{A}_x}$. We wish to define a nominal family $A \multimap_X B = (C_x)_{x \in X}$.

From Section 6, we already have a monoidal structure $\otimes_x$ on each fiber $\mathbf{Nom}_{\mathbb{A}_x}$. Moreover, since this monoidal structure was induced by some isomorphism with $\mathbf{Nom}$, it is closed. Let $\multimap_x$ be the internal hom-functor on $\mathbf{Nom}_{\mathbb{A}_x}$. Then we can define the family of objects C_x as follows:

$$C_x = A_x \multimap_x B_x.$$

To make this into a nominal family, we must define a family of equivariant maps $\varphi^C_{\pi,x} : \widehat{\pi}(C_x) \to C_{\pi \bullet x}$, satisfying properties (a) and (b) of Remark 5.4. We define these as follows. Here $y = \pi \bullet x$ as usual, and $n_\pi : \widehat{\pi}(A_x \multimap_x B_x) \to \widehat{\pi}(A_x) \multimap_y \widehat{\pi}(B_x)$ is the map induced from $m_\pi : \widehat{\pi}(A_x \otimes_x B_x) \to \widehat{\pi}(A_x) \otimes_y \widehat{\pi}(B_x)$ as in Section 7.1.

$$\begin{tikzcd}
\widehat{\pi}(A_x \multimap_x B_x) \arrow[rr, "\varphi^{C}_{\pi,x}"] \arrow[dr, "n_\pi"'] & & A_y \multimap_y B_y \\
& \widehat{\pi}(A_x) \multimap_y \widehat{\pi}(B_x) \arrow[ur, "(\varphi^{A}_{\pi,x})^{-1} \multimap_y \varphi^{B}_{\pi,x}"'] &
\end{tikzcd}$$

Properties (a) and (b) of Remark 5.4 are shown in Appendix D. This finishes the proof that $(C_x, \varphi^{C}_{\pi,x})$ is a nominal family, i.e., that $A \multimap_X B$ is a well-defined object of $\mathbf{Nom}^X$.

7.2.2 Definition of $f \multimap_X g$ and functoriality

To continue our definition of the closed structure on $\mathbf{Nom}/X$, we must now define the morphism part of the functor $\multimap_X$. Therefore, consider $f : A' \to A$ and $g : B \to B'$ in $\mathbf{Nom}/X$. We will define a morphism $f \multimap_X g : A \multimap_X B \to A' \multimap_X B'$ as a family of maps $(f \multimap_X g)x : (A \multimap_X B)_x \to (A' \multimap_X B')_x$. They are defined by

$$\begin{tikzcd}
(A \multimap_X B)_x \arrow[r, "(f \multimap_X g)_x"] \arrow[d, equal] & (A' \multimap_X B')_x \arrow[d, equal] \\
A_x \multimap_x B_x \arrow[r, "f_x \multimap_x g_x"] & A'_x \multimap_x B'_x.
\end{tikzcd}$$

To make sure that this is a well-defined morphism of nominal families, by Remark 5.4, it is sufficient to show that these maps are compatible with the φ-maps, i.e., that the outer part of the following diagram commutes for all $\pi \in \Pi_{\mathbb{A}}$, where $y = \pi \bullet x$.

$$\begin{tikzcd}
\widehat{\pi}(A_x \multimap_x B_x) \arrow[rrrrr, "\widehat{\pi}(f_x \multimap_x g_x)"] \arrow[ddddd, "\varphi^{A \multimap_X B}_{\pi,x}"'] \arrow[ddrr, "n_\pi"] & & & & & \widehat{\pi}(A'_x \multimap_x B'_x) \arrow[ddll, "n_\pi"'] \arrow[ddddd, "\varphi^{A' \multimap_X B'}_{\pi,x}"] \\
& & {} \arrow[r, phantom, "(1)"] & {} & & \\
& (2) & \widehat{\pi}(A_x) \multimap_y \widehat{\pi}(B_x) \arrow[r, "\widehat{\pi}(f_x) \multimap_y \widehat{\pi}(g_x)"] \arrow[ddll, "(\varphi^{A}_{\pi,x})^{-1} \multimap_y \varphi^{B}_{\pi,x}"] & \widehat{\pi}(A'_x) \multimap_y \widehat{\pi}(B'_x) \arrow[ddrr, "(\varphi^{A'}_{\pi,x})^{-1} \multimap_y \varphi^{B'}_{\pi,x}"'] & (2) & \\
& & {} \arrow[r, phantom, "(3)"] & {} & & \\
& & & & & \\
A_y \multimap_y B_y \arrow[rrrrr, "f_y \multimap_y g_y"] & & & & & A'_y \multimap_y B'_y.
\end{tikzcd}$$

Part (1) commutes because n_π is a natural transformation; the parts labelled (2) commute by definition of the φ-maps for $\multimap_X$; part (3) commutes since f and g are morphisms of nominal families, and are therefore compatible with their respective φ-maps. This finishes the proof that $f \multimap_X g : A \multimap_X B \to A' \multimap_X B'$ is a well-defined morphism.

Next, we must show that $\multimap_X$ is a functor. However, this is straightforward because the required equations hold fiberwise: we have $\mathrm{id}^{A}_{x} \multimap_x \mathrm{id}^{B}_{x} = \mathrm{id}^{A \multimap B}_{x}$ and $(f_x \multimap_x g_x) \circ (f'_x \multimap_x g'_x) = (f'_x \circ f_x) \multimap_x (g_x \circ g'_x)$, because $\multimap_x$ is a (mixed variance) functor on $\mathbf{Nom}_{\mathbb{A}_x}$ for every $x \in X$.

7.2.3 Adjunction between $\multimap_X$ and $\otimes_X$

To finish our definition of the closed structure on $\mathbf{Nom}/X$, we must show that the functor $B \multimap_X (-)$ is a right adjoint of $(-) \otimes_X B$. Most of the proof is fiberwise, i.e., we already know that the functor $B_x \multimap_x (-)$ is a right adjoint of $(-) \otimes_x B_x$ in every fiber $x \in X$. The only additional thing we must prove is that the fiberwise adjunction respects morphisms of nominal families, i.e., that if $f_x : A_x \otimes_x B_x \to C_x$ and $g_x : A_x \to B_x \multimap_x C_x$ are fiberwise adjoint mates, then the family $(f_x)_{x \in X}$ respects the φ-maps if and only if the family $(g_x)_{x \in X}$ does.

So consider a family of maps $f_x : A_x \otimes_x B_x \to C_x$, and let $g_x : A_x \to B_x \multimap_x C_x$ be the curry of f_x for every $x \in X$. We must show that the perimeter of the left diagram commutes if and only if the perimeter of the right one does:

$$\begin{tikzcd}
\widehat{\pi}(A_x \otimes_x B_x) \arrow[r, "\widehat{\pi}(f_x)"] \arrow[d, "m_\pi"] \arrow[d, phantom, shift right=8ex, "(0)"] \arrow[dd, bend right=60, "\varphi^{A\otimes_x B}_{\pi,x}"'] & \widehat{\pi}(C_x) \arrow[dd, "\varphi^{C}_{\pi,x}"] \\
\widehat{\pi}(A_x) \otimes_{\pi\bullet x} \widehat{\pi}(B_x) \arrow[ur, "h"'] \arrow[ur, phantom, shift left=3ex, "(1)"] \arrow[d, "\varphi^{A}_{\pi,x} \otimes_{\pi\bullet x} \varphi^{B}_{\pi,x}"] & (2) \\
A_{\pi\bullet x} \otimes_{\pi\bullet x} B_{\pi\bullet x} \arrow[r, "f_{\pi\bullet x}"'] & C_{\pi\bullet x}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\pi}(A_x) \arrow[r, "\widehat{\pi}(g_x)"] \arrow[dd, "\varphi^{A}_{\pi,x}"'] \arrow[dr, "h^c"'] & \widehat{\pi}(B_x \multimap_x C_x) \arrow[d, "n_\pi"'] \arrow[d, phantom, shift right=5ex, "(4)"] \arrow[d, phantom, shift left=6ex, "(5)"] \arrow[dd, bend left=60, "\varphi^{B\multimap_x C}_{\pi,x}"] \\
(3) & \widehat{\pi}(B_x) \multimap_{\pi\bullet x} \widehat{\pi}(C_x) \arrow[d, "\left(\varphi^{B}_{\pi,x}\right)^{-1} \multimap_{\pi\bullet x} \varphi^{C}_{\pi,x}"'] \\
A_{\pi\bullet x} \arrow[r, "g_{\pi\bullet x}"'] & B_{\pi\bullet x} \multimap_{\pi\bullet x} C_{\pi\bullet x}
\end{tikzcd}$$

To see why, first define h to be the unique map such that (1) commutes. Let h^c be the result of currying h; then (4) commutes by the definition of n and the fact that g_x if the curried form of f_x. Parts (0) and (5) commute by the definitions of $\varphi^{A\otimes_x B}_{\pi,x}$ and $\varphi^{B\multimap_x C}_{\pi,x}$, respectively. Then by the naturality of currying, (2) commutes if and only if (3) commutes, which implies that the perimeter of the left diagram commutes if and only if the perimeter of the right one does, as claimed.

This finishes the proof of Theorem 7.1.

8 Conclusion and future work

We showed that every monoidal (closed) structure on **Nom** induces a corresponding monoidal (closed) structure on all of its slice categories $\mathbf{Nom}/X$. It seems reasonable to say that **Nom** is therefore "locally monoidal closed". However, as far as we know, there is no general definition of local monoidal structure: given a general category $\mathbf{C}$, it may not in general make sense to ask whether a monoidal structure on $\mathbf{C}/X$ is "induced" by a monoidal structure on $\mathbf{C}$. It is an interesting question what requirements a category $\mathbf{C}$ should satisfy so that one can speak of locally monoidal and locally monoidal closed structures.

A possible limitation of our result is that there are only a few interesting examples of monoidal closed structure on **Nom**. The best-known ones are the cartesian structure and the separated one; another (non-symmetric) monoidal closed structure was very recently discovered by [10]. For each of these cases, the corresponding monoidal closed structures on the slice categories could have been defined in elementary terms, yielding a more concrete element-wise proof. Nevertheless, we believe there is some value in having proved this result in full generality. For one, it is pleasing that the proof is uniform, i.e., it works identically regardless of the particular monoidal closed structure in question. Second, part of our result is not just for monoidal closed structures, but more generally for monoidal ones, of which there are many. Third, the technique used in this paper is clearly not limited to just monoidal structures, but can be expected to work without much difficulty for other similar categorical structures, i.e., those that are given by functors, natural transformations, and equations. Thus, we expect that there is a much larger class of structures that can be lifted from **Nom** to its slice categories. The details of this are left for future work.

As mentioned in the introduction, one of the motivations for studying the structure of slice categories on **Nom** is that slice categories play an important role in models of dependent type theory. We hope that the present work can serve as a building block towards the eventual goal of constructing a model for a nominal dependent type theory.

References

[1] *Agda documentation*, `https://agda.readthedocs.io/en/latest/`. Accessed: 2026-02-12.

[2] Cheney, J., *A dependent nominal type theory*, Log. Methods Comput. Sci. **8** (2012).

[3] Clairambault, P. and P. Dybjer, *The biequivalence of locally cartesian closed categories and Martin-Löf type theories*, Mathematical Structures in Computer Science **24** (2014), ISSN 1469-8072.
`https://doi.org/10.1017/s0960129513000881`

[4] Clouston, R., *Generalised name abstraction for nominal sets*, in: *Proceedings of the 16th International Conference on Foundations of Software Science and Computation Structures, FOSSACS 2013*, Lecture Notes in Computer Science 7794, pages 434–449 (2013).

[5] de Moura, L. and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: *Proceedings of the 28th International Conference on Automated Deduction (CADE 28), July 12–15, 2021*, pages 625–635, Springer-Verlag (2021).
`https://doi.org/10.1007/978-3-030-79876-5_37`

[6] Fairweather, E., M. Fernández, N. Szasz and A. Tasistro, *Dependent Types for Nominal Terms with Atom Substitutions*, in: *13th International Conference on Typed Lambda Calculi and Applications (TLCA 2015)*, volume 38 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 180–195, Schloss Dagstuhl – Leibniz-Zentrum für Informatik (2015).

[7] Gabbay, M. J. and A. Pitts, *A new approach to abstract syntax involving binders*, in: *Proceedings of the 14th Annual IEEE Symposium on Logic in Computer Science*, LICS '99, page 214, IEEE Computer Society, USA (1999).

[8] Gabbay, M. J. and A. M. Pitts, *A new approach to abstract syntax with variable binding*, Formal Aspects of Computing **13**, pages 341–363 (2002).

[9] Gadducci, F., M. Miculan and U. Montanari, *About permutation algebras, (pre)sheaves and named sets*, Higher-Order and Symbolic Computation **19**, pages 283–304 (2006).
`https://doi.org/10.1007/s10990-006-8749-3`

[10] Lenke, F., S. Milius and H. Urbat, *A unified treatment of substitution for presheaves, nominal sets, renaming sets, and so on* (2026). Available from `arXiv:2602.11907`.

[11] Milius, S., L. Schröder and T. Wißmann, *Regular behaviours with names* (2016). Available from `arXiv:1607.07828`.

[12] Pitts, A. M., *Nominal Sets: Names and Symmetry in Computer Science*, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press (2013).

[13] Pitts, A. M., J. Matthiesen and J. Derikx, *A dependent type theory with abstractable names*, in: *Workshop on Logical and Semantic Frameworks with Applications* (2015).

[14] *Rocq documentation*, `https://rocq-prover.org/refman`. Accessed: 2026-02-12.

[15] Seely, R. A. G., *Locally cartesian closed categories and type theory*, Mathematical Proceedings of the Cambridge Philosophical Society **95**, pages 33–48 (1984).

[16] Van Muylder, A., A. Nuyts and D. Devriese, *Nominal type theory by nullary internal parametricity* (2025). Available from `arXiv:2512.09464`.

A Some proofs

Proof of Proposition 3.2. Let $\sigma \in \Pi_{\text{large}}$ and $x \in X$. We would like to define $\sigma \mathbin{\hat{\bullet}} x = \pi \bullet x$, where π is some finitely supported permutation such that $\pi|_{\text{supp}(x)} = \sigma|_{\text{supp}(x)}$. For this to work, we must show that:

(i) Such a permutation π exists. Indeed, we know $\text{supp}(x)$ is finite. Let $\text{supp}(x) = \{a_1, a_2, \ldots, a_n\}$. Let $b_i = \sigma(a_i)$ for $i = 1, \ldots, n$. Let $U = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$, which may have any number of elements between n and $2n$, say $n + k$ elements. Let $\{c_1, c_2, \ldots, c_k\} = U - \{a_1, a_2, \ldots, a_n\}$ and $\{d_1, d_2, \ldots, d_k\} = U - \{b_1, b_2, \ldots, b_n\}$, and define $\pi \in \Pi$ by

$$\pi(a_i) = b_i \quad \text{and} \quad \pi(c_j) = d_j$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Also, by definition $\pi|_{\text{supp}(x)} = \sigma|_{\text{supp}(x)}$.

(ii) The choice of π is not important, i.e., if π and π' satisfy $\pi|_{\text{supp}(x)} = \pi'|_{\text{supp}(x)} = \sigma|_{\text{supp}(x)}$, then $\pi \bullet x = \pi' \bullet x$. This holds by Lemma 3.1. Hence, $\hat{\bullet} : \Pi_{\text{large}} \times X \to X$ is a well-defined operation. We also note that if σ is finitely supported, we have $\sigma \mathbin{\hat{\bullet}} x = \sigma \bullet x$.

(iii) The operation $\hat{\bullet}$ is an action, i.e., $\mathrm{id} \mathbin{\hat{\bullet}} x = x$ and for all $\sigma_1, \sigma_2 \in \Pi_{\text{large}}$, $(\sigma_2\sigma_1) \mathbin{\hat{\bullet}} x = \sigma_2 \mathbin{\hat{\bullet}} (\sigma_1 \mathbin{\hat{\bullet}} x)$. For the first claim, we clearly have $\mathrm{id} \mathbin{\hat{\bullet}} x = \mathrm{id} \bullet x = x$. For the second claim, choose some $\pi_1, \pi_2 \in \Pi$ such that

$$\pi_1|_{\mathrm{supp}(x)} = \sigma_1|_{\mathrm{supp}(x)} \tag{A.1}$$

and

$$\pi_2|_{\mathrm{supp}(\pi_1 \bullet x)} = \sigma_2|_{\mathrm{supp}(\pi_1 \bullet x)}. \tag{A.2}$$

We first claim that $\pi_2\pi_1|_{\mathrm{supp}(x)} = \sigma_2\sigma_1|_{\mathrm{supp}(x)}$. Indeed, for $a \in \mathrm{supp}(x)$, we have $\sigma_2\sigma_1(a) = \sigma_2(\sigma_1(a)) = \sigma_2(\pi_1(a)) = \pi_2(\pi_1(a)) = \pi_2\pi_1(a)$, where the second and third equations hold by (A.1) and (A.2), respectively. It follows that $(\sigma_1\sigma_2) \mathbin{\hat{\bullet}} x = (\pi_1\pi_2) \bullet x$. Then we have

$$\begin{aligned}
\sigma_2 \mathbin{\hat{\bullet}} (\sigma_1 \mathbin{\hat{\bullet}} x) &= \sigma_2 \mathbin{\hat{\bullet}} (\pi_1 \bullet x) \\
&= \pi_2 \bullet (\pi_1 \bullet x) \\
&= (\pi_2\pi_1) \bullet x \\
&= (\sigma_2\sigma_1) \mathbin{\hat{\bullet}} x,
\end{aligned}$$

as claimed. Thus, the operation $\hat{\bullet}$ is a valid action.

□

Proof of Proposition 4.5. We first need to show that $\sigma \mathbin{\hat{\bullet}} (-) : \widehat{\sigma}(X) \to X$ is a well-defined morphism, i.e., that it is equivariant. So consider any $\pi \in \Pi_{\mathbb{A}}$ and $x \in X$. We have

$$\sigma \mathbin{\hat{\bullet}} (\pi \diamond x) = \sigma \mathbin{\hat{\bullet}} ((\sigma^{-1}\pi\sigma) \bullet x) = \sigma \mathbin{\hat{\bullet}} \sigma^{-1} \mathbin{\hat{\bullet}} \pi \mathbin{\hat{\bullet}} \sigma \mathbin{\hat{\bullet}} x = \pi \bullet (\sigma \mathbin{\hat{\bullet}} x).$$

Next, we need to show naturality. So consider objects $(X, \bullet)$, $(Y, \diamond) \in \mathbf{Nom}_{\mathbb{A}}$. Here, for clarity, we have used two different symbols for the actions on X and Y. Let $f : X \to Y$ be an equivariant map. We must show that the following square commutes:

$$\begin{tikzcd}
\widehat{\sigma}(X) \arrow[r, "\widehat{\sigma}(f)"] \arrow[d, "\sigma \mathbin{\hat{\bullet}} (-)"'] & \widehat{\sigma}(Y) \arrow[d, "\sigma \mathbin{\hat{\diamond}} (-)"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

In other words, for all $x \in X$, we need to show that $f(\sigma \mathbin{\hat{\bullet}} x) = \sigma \mathbin{\hat{\diamond}} f(x)$. Let π be a finitely supported permutation such that π and σ agree on the $\mathrm{supp}(x)$. Note that $\mathrm{supp}(f(x)) \subseteq \mathrm{supp}(x)$, so π and σ agree on $\mathrm{supp}(f(x))$ as well. Then, because f is equivariant, we have

$$f(\sigma \mathbin{\hat{\bullet}} x) = f(\pi \bullet x) = \pi \diamond f(x) = \sigma \mathbin{\hat{\diamond}} f(x).$$

Finally, the map $\sigma \mathbin{\hat{\bullet}} (-)$ is clearly invertible with inverse $\sigma^{-1} \mathbin{\hat{\bullet}} (-)$. □

Proof of Lemma 6.4. Define $\sigma_1, \sigma_2, \sigma_3 : \mathbb{B} \to \mathbb{B}$ by $\sigma_1 = \gamma_2 \circ (\gamma_1)^{-1}$, $\sigma_2 = \gamma_3 \circ (\gamma_2)^{-1}$, and $\sigma_3 = \gamma_3 \circ (\gamma_1)^{-1}$. Note that $\sigma_3 = \sigma_2\sigma_1$. Consider $\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3 : \mathbf{Nom}_{\mathbb{B}} \to \mathbf{Nom}_{\mathbb{B}}$ as in Section 4.2. Consider the following diagram:

$$\begin{tikzcd}
 & & \widehat{\sigma_3}(A \otimes_1 B) \arrow[rr, equal, bend left=10] \arrow[dl, equal] & & \widehat{\sigma_3}(A) \otimes_3 \widehat{\sigma_3}(B) \arrow[dr, equal] & \\
 & \widehat{\sigma_2}(\widehat{\sigma_1}(A \otimes_1 B)) \arrow[rr, equal] \arrow[d, "\sigma_2 \hat{\diamond}(-)", "\cong"'] \arrow[dd, bend right=60, "\sigma_3 \hat{\bullet}(-)"', "\cong"] & & \widehat{\sigma_2}(\widehat{\sigma_1}(A) \otimes_2 \widehat{\sigma_1}(B)) \arrow[rr, equal] \arrow[dl, "\sigma_2 \hat{\diamond}(-)"', "\cong"] \arrow[dr, "\widehat{\sigma_2}(\sigma_1 \hat{\bullet}(-) \otimes_2 \sigma_1 \hat{\bullet}(-))"', "\cong"] & & \widehat{\sigma_2}(\widehat{\sigma_1}(A)) \otimes_3 \widehat{\sigma_2}(\widehat{\sigma_1}(B)) \arrow[d, "\cong", "\widehat{\sigma_2}(\sigma_1 \hat{\bullet}(-)) \otimes_3 \widehat{\sigma_2}(\sigma_1 \hat{\bullet}(-))"'] \arrow[dd, bend left=60, "\sigma_3 \hat{\bullet}(-) \otimes_3 \sigma_3 \hat{\bullet}(-)", "\cong"'] \\
 & \widehat{\sigma_1}(A \otimes_1 B) \arrow[r, equal] \arrow[d, "\sigma_1 \hat{\bullet}(-)", "\cong"'] & \widehat{\sigma_1}(A) \otimes_2 \widehat{\sigma_1}(B) \arrow[dr, "\sigma_1 \hat{\bullet}(-) \otimes_2 \sigma_1 \hat{\bullet}(-)", "\cong"'] & & \widehat{\sigma_2}(A \otimes_2 B) \arrow[r, equal] \arrow[dl, "\sigma_2 \hat{\bullet}(-)", "\cong"'] & \widehat{\sigma_2}(A) \otimes_3 \widehat{\sigma_2}(B) \arrow[d, "\cong", "\sigma_2 \hat{\bullet}(-) \otimes_3 \sigma_2 \hat{\bullet}(-)"'] \\
 & A \otimes_1 B \arrow[rr, "\eta^{A,B}_{\gamma_1,\gamma_2}"'] \arrow[rrrr, bend right=20, "\eta^{A,B}_{\gamma_1,\gamma_3}"'] & & A \otimes_2 B \arrow[rr, "\eta^{A,B}_{\gamma_2,\gamma_3}"'] & & A \otimes_3 B
\arrow[from=2-2, to=3-3, phantom, "(5)"]
\arrow[from=2-4, to=3-6, phantom, "(6)"]
\arrow[from=3-3, to=3-5, phantom, "(7)"]
\arrow[from=3-2, to=4-4, phantom, "(1)"]
\arrow[from=3-6, to=4-4, phantom, "(2)"]
\arrow[from=2-2, to=4-2, phantom, bend right=30, "(3)"]
\arrow[from=2-6, to=4-6, phantom, bend left=30, "(4)"]
\arrow[from=4-2, to=4-6, phantom, bend right=10, "(8)"]
\end{tikzcd}$$

We must show that (8) holds. Note that (1), (2), and the perimeter commute by Remark 6.3. (3) and (4) commute by Lemma 4.6. (5) and (6) commute because $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ strictly preserve the relevant monoidal structures, and (7) commutes by naturality of $\sigma_2 \hat{\bullet}(-)$. Since all maps in this diagram are isomorphisms, it follows that (8) commutes, as claimed. ☐

Proof of Lemma 6.5. Let $\sigma = \gamma_2\gamma_1^{-1}$. Note that $(\pi\gamma_2)(\pi\gamma_1)^{-1} = \pi\sigma\pi^{-1}$. For clarity, we use “$\bullet$” to denote all of the actions in the category $\mathbf{Nom}_{\mathbb{A}}$ and “$\diamond$” for the actions in $\mathbf{Nom}_{\mathbb{B}}$. Consider the following diagram.

$$\begin{tikzcd}
 & & \widehat{\pi}(\widehat{\sigma}(A \otimes_1 B)) \arrow[r, equal] \arrow[d, "\widehat{\pi}(\sigma \hat{\bullet}(-))"'] \arrow[dll, equal, bend right=20] & \widehat{\pi}(\widehat{\sigma}(A) \otimes_2 \widehat{\sigma}(B)) \arrow[d, "\widehat{\pi}(\sigma \hat{\bullet}(-) \otimes_2 \sigma \hat{\bullet}(-))"'] \arrow[dr, equal, bend left=20] & \\
\widehat{\pi\sigma}(A \otimes_1 B) \arrow[d, equal] & & \widehat{\pi}(A \otimes_1 B) \arrow[r, "\widehat{\pi}(\eta^{A,B}_{\gamma_1,\gamma_2})"'] \arrow[d, equal] & \widehat{\pi}(A \otimes_2 B) \arrow[d, equal] & \widehat{\pi}(\widehat{\sigma}(A)) \otimes_2' \widehat{\pi}(\widehat{\sigma}(B)) \arrow[dl, "\widehat{\pi}(\sigma \hat{\bullet}(-)) \otimes_2' \widehat{\pi}(\sigma \hat{\bullet}(-))"'] \arrow[d, equal] \\
\widehat{\pi\sigma\pi^{-1}}(\widehat{\pi}(A \otimes_1 B)) \arrow[urr, "\pi\sigma\pi^{-1} \hat{\diamond}(-)"] \arrow[dr, equal] & & \widehat{\pi}(A) \otimes_1' \widehat{\pi}(B) \arrow[r, "\eta^{A,B}_{\pi\gamma_1,\pi\gamma_2}"'] & \widehat{\pi}(A) \otimes_2' \widehat{\pi}(B) & \widehat{\pi\sigma}(A) \otimes_2' \widehat{\pi\sigma}(B) \arrow[d, equal] \\
 & \widehat{\pi\sigma\pi^{-1}}(\widehat{\pi}(A) \otimes_1' \widehat{\pi}(B)) \arrow[ur, "\pi\sigma\pi^{-1} \hat{\diamond}(-)"'] \arrow[rrr, equal] & & & \widehat{\pi\sigma\pi^{-1}}(\widehat{\pi}(A)) \otimes_2' \widehat{\pi\sigma\pi^{-1}}(\widehat{\pi}(B)) \arrow[ul, "\pi\sigma\pi^{-1} \hat{\diamond}(-) \otimes_2' \pi\sigma\pi^{-1} \hat{\diamond}(-)"']
\arrow[from=1-3, to=2-4, phantom, "(2)"]
\arrow[from=2-1, to=1-3, phantom, "(5)"]
\arrow[from=1-4, to=2-5, phantom, "(4)"]
\arrow[from=2-3, to=3-4, phantom, "(1)"]
\arrow[from=3-1, to=3-3, phantom, "(3)"]
\arrow[from=3-3, to=4-4, phantom, "(2)"]
\arrow[from=3-4, to=3-5, phantom, "(5)"]
\end{tikzcd}$$

We must show that (1) commutes. The parts labelled (2) commute by definition of $\eta^{A,B}$ map, (3) commutes by equality of nominal sets, (4) commutes since $\widehat{\pi}$ is a monoidal functor, and the two parts labelled (5) commute by Lemma 4.7. ☐

Proof of Lemma 6.6. On $\mathbf{Nom}_{\mathbb{C}}$, let $\otimes_2'$ be the monoidal structure induced by $\mu_1\delta_1$. On $\mathbf{Nom}_{\mathbb{D}}$, let $\otimes_3'$ and $\otimes_3''$ be the monoidal structures induced by $\mu_2\delta_2$ and $\mu_2\mu_1\delta_1$, respectively. Consider the following diagram.

$$\begin{tikzcd}
 & & \widehat{\mu_1}(\widehat{\mu_2}(A)\otimes_2\widehat{\mu_2}(B)) \arrow[ddrr, equal] \arrow[dddrrrr, bend left, "m_{\mu_1}"] & & & & \\
 & (1) & & & (1) & & \\
 & \widehat{\mu_1}(\widehat{\mu_2}(A)\otimes'_2\widehat{\mu_2}(B)) \arrow[uur, "\widehat{\mu_1}(\eta_{\mu_2\delta_1,\delta_2})"] \arrow[dr, equal] & (2) & & \widehat{\mu_1}\widehat{\mu_2}(A)\otimes'_3\widehat{\mu_1}\widehat{\mu_2}(B) \arrow[drr, "\eta_{\mu_1\delta_2,\delta_3}"] \arrow[dddd, equal] & & \\
\widehat{\mu_1}\widehat{\mu_2}(A\otimes_1 B) \arrow[uuurr, bend left, "\widehat{\mu}_1(m_{\mu_2})"] \arrow[ur, equal] \arrow[dddd, equal] & & \widehat{\mu_1}\widehat{\mu_2}(A)\otimes''_3\widehat{\mu_1}\widehat{\mu_2}(B) \arrow[urr, "\eta_{\mu_1\mu_2\delta_1,\mu_1\delta_2}"] \arrow[dddd, equal] & & & & \widehat{\mu_1}\widehat{\mu_2}(A)\otimes_3\widehat{\mu_1}\widehat{\mu_2}(B) \arrow[dddd, equal] \\
 & (5) & & (4) & & (4) & \\
 & & & & \widehat{\mu_1\mu_2}(A)\otimes'_3\widehat{\mu_1\mu_2}(B) \arrow[ddrr, "\eta_{\mu_1\delta_2,\delta_3}"] & & \\
 & & & & (3) & & \\
\widehat{\mu_1\mu_2}(A\otimes_1 B) \arrow[rr, equal] \arrow[rrrrrr, bend right, "m_{\mu_1\mu_2}"'] & & \widehat{\mu_1\mu_2}(A)\otimes''_3\widehat{\mu_1\mu_2}(B) \arrow[uurr, "\eta_{\mu_1\mu_2\delta_1,\mu_1\delta_2}"] \arrow[rrrr, "\eta_{\mu_1\mu_2\delta_1,\delta_3}"] & & & & \widehat{\mu_1\mu_2}(A)\otimes_3\widehat{\mu_1\mu_2}(B) \\
 & & (1) & & & &
\end{tikzcd}$$

We must show that the outside commutes. Each part labelled (1) commutes by definition of the m-maps; (2) commutes by Lemma 6.5; (3) commutes by Lemma 6.4; each part labelled (4) commutes by naturality of η; and (5) commutes because all its bounding morphisms are equalities. ☐

Proof of Lemma 6.7. Consider the following diagram, where $\sigma = \delta_2(\mu\delta_1)^{-1} = \mu^{-1}$.

$$\begin{tikzcd}
 & & (1) & & \\
\widehat{\mu}(A\otimes_1 B) \arrow[rr, equal] \arrow[rrrr, bend left, "m_\mu"] & & \widehat{\mu}(A)\otimes'_2\widehat{\mu}(B) \arrow[rr, "\eta_{\mu\delta_1,\delta_2}"] & & \widehat{\mu}(A)\otimes_2\widehat{\mu}(B) \\
 & (3) & & (2) & \\
\widehat{\sigma}\widehat{\mu}(A\otimes_1 B) \arrow[uu, "\sigma\mathbin{\hat{\bullet}}(-)"] \arrow[rr, equal] \arrow[d, equal] & & \widehat{\sigma}(\widehat{\mu}(A)\otimes'_2\widehat{\mu}(B)) \arrow[uu, "\sigma\mathbin{\hat{\bullet}}(-)"] \arrow[rr, equal] & & \widehat{\sigma}\widehat{\mu}(A)\otimes_2\widehat{\sigma}\widehat{\mu}(B) \arrow[uu, "\sigma\mathbin{\hat{\bullet}}(-)\otimes_2\sigma\mathbin{\hat{\bullet}}(-)"'] \arrow[d, equal] \\
A\otimes_1 B \arrow[rrrr, equal] & & & & A\otimes_2 B.
\end{tikzcd}$$

The part labelled (1) commutes by definition of m_μ. The part labelled (2) commutes by Remark 6.3. The part labelled (3) commutes because the left and right objects are literally the same nominal set. Therefore, the outside commutes. This implies the Lemma because $\mu = \sigma^{-1}$. ☐

B Proof of properties (a) and (b) from Section 6.4.1

Lemma B.1. *The family of maps $\varphi^C_{\pi,x}$ defined in Section 6.4.1 satisfies properties (a) and (b) of Remark 5.4.*

Proof. We first need to show that $\sigma\mathbin{\hat{\bullet}}(-) : \widehat{\sigma}(X) \to X$ is a well-defined morphism, i.e., that it is equivariant. So consider any $\pi \in \Pi_{\mathbb{A}}$ and $x \in X$. We have

$$\sigma\mathbin{\hat{\bullet}}(\pi\diamond x) = \sigma\mathbin{\hat{\bullet}}((\sigma^{-1}\pi\sigma)\bullet x) = \sigma\mathbin{\hat{\bullet}}\sigma^{-1}\mathbin{\hat{\bullet}}\pi\mathbin{\hat{\bullet}}\sigma\mathbin{\hat{\bullet}}x = \pi\bullet(\sigma\mathbin{\hat{\bullet}}x).$$

Next, we need to show naturality. So consider objects $(X, \bullet)$, $(Y, \diamond) \in \mathbf{Nom}_{\mathbb{A}}$. Here, for clarity, we have used two different symbols for the actions on X and Y. Let $f : X \to Y$ be an equivariant map. We must show that the following square commutes:

$$\begin{tikzcd}
\widehat{\sigma}(X) \arrow[r, "\widehat{\sigma}(f)"] \arrow[d, "\sigma \mathbin{\hat{\bullet}} (-)"'] & \widehat{\sigma}(Y) \arrow[d, "\sigma \mathbin{\hat{\diamond}} (-)"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

In other words, for all $x \in X$, we need to show that $f(\sigma \mathbin{\hat{\bullet}} x) = \sigma \mathbin{\hat{\diamond}} f(x)$. Let π be a finitely supported permutation such that π and σ agree on the $\operatorname{supp}(x)$. Note that $\operatorname{supp}(f(x)) \subseteq \operatorname{supp}(x)$, so π and σ agree on $\operatorname{supp}(f(x))$ as well. Then we have

$$f(\sigma \mathbin{\hat{\bullet}} x) = f(\pi \bullet x) = \pi \diamond f(x) = \sigma \mathbin{\hat{\diamond}} f(x).$$

Finally, the map $\sigma \mathbin{\hat{\bullet}} (-)$ is clearly invertible with inverse $\sigma^{-1} \mathbin{\hat{\bullet}} (-)$. $\square$

C Proof of compatibility of the associators and unitors from Section 6.4.3

Recall that we defined

$$\begin{aligned}
\alpha_x^{A,B,C} &: (A_x \otimes_x B_x) \otimes_x C_x \to A_x \otimes_x (B_x \otimes_x C_x), \\
\lambda_x^A &: I_x \otimes_x A_x \to A_x, \\
\rho_x^A &: A_x \otimes_x I_x \to A_x.
\end{aligned}$$

To show that these maps are morphisms of nominal families, by Remark 5.4, we must show that the following diagrams commute, where $y = \pi \bullet x$:

$$\begin{tikzcd}
\widehat{\pi}((A_x \otimes_x B_x) \otimes_x C_x) \arrow[r, "\widehat{\pi}(\alpha_x)"] \arrow[d, "\varphi_{\pi,x}^{(A\otimes_X B)\otimes_X C}"'] & \widehat{\pi}(A_x \otimes_x (B_x \otimes_x C_x)) \arrow[d, "\varphi_{\pi,x}^{A\otimes_X (B\otimes_X C)}"] \\
(A_y \otimes_y B_y) \otimes_y C_y \arrow[r, "\alpha_y"] & A_y \otimes_y (B_y \otimes_y C_y)
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{\pi}(A_x) \arrow[r, "\widehat{\pi}(\lambda_x)"] \arrow[d, "\varphi_{\pi,x}^{A}"'] & \widehat{\pi}(I_x \otimes_x A_x) \arrow[d, "\varphi_{\pi,x}^{I\otimes_X A}"] \\
A_y \arrow[r, "\lambda_y"] & I_y \otimes_y A_y
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\pi}(A_x) \arrow[r, "\widehat{\pi}(\rho_x)"] \arrow[d, "\varphi_{\pi,x}^{A}"'] & \widehat{\pi}(A_x \otimes_x I_x) \arrow[d, "\varphi_{\pi,x}^{A\otimes_X I}"] \\
A_y \arrow[r, "\rho_y"] & A_y \otimes_y I_y
\end{tikzcd}$$

We only prove the diagram for α, as the other two are very similar. Consider the following:

$$\begin{tikzcd}
\widehat{\pi}((A_x \otimes_x B_x) \otimes_x C_x) \arrow[r, "\widehat{\pi}(\alpha_x)"] \arrow[d, "m_\pi^3"] \arrow[dd, bend right=60, "\varphi_{\pi,x}^{(A\otimes_X B)\otimes_X C}"'] & \widehat{\pi}(A_x \otimes_x (B_x \otimes_x C_x)) \arrow[d, "m_\pi^{3'}"'] \arrow[dd, bend left=60, "\varphi_{\pi,x}^{A\otimes_X (B\otimes_X C)}"] \\
(\widehat{\pi}(A_x) \otimes_y \widehat{\pi}(B_x)) \otimes_y \widehat{\pi}(C_x) \arrow[r, "\alpha_y"] \arrow[d, "(\varphi_{\pi,x}^A \otimes_y \varphi_{\pi,x}^B) \otimes_y \varphi_{\pi,x}^C"] & \widehat{\pi}(A_x) \otimes_y (\widehat{\pi}(B_x) \otimes_y \widehat{\pi}(C_x)) \arrow[d, "\varphi_{\pi,x}^A \otimes_y (\varphi_{\pi,x}^B \otimes_y \varphi_{\pi,x}^C)"'] \\
(A_y \otimes_y B_y) \otimes_y C_y \arrow[r, "\alpha_y"] & A_y \otimes_y (B_y \otimes_y C_y)
\end{tikzcd}$$

Here, the maps m_π^3 and $m_\pi^{3'}$ are the obvious morphisms defined by using the m-maps twice. The parts labelled (1) commute by definition of $\varphi_{\pi,x}^{(A\otimes_X B)\otimes_X C}$ and $\varphi_{\pi,x}^{A\otimes_X (B\otimes_X C)}$. The part labelled (2) commutes because m_π is the structure that makes $\widehat{\pi}$ into a monoidal functor. The part labelled (3) commutes by naturality of α_y.

D Proof of Properties (a) and (b) from Section 7.2.1

Lemma D.1. *The family of maps $\varphi_{\pi,x}^C$ defined in Section 7.2.1 satisfies properties (a) and (b) of Remark 5.4.*

Proof. We must prove properties (a) and (b) of Remark 5.4. For (a), consider any $\pi_1, \pi_2 \in \Pi$ and some $x \in X$. We must show that $\varphi_{\pi_1\pi_2,x}^C = \varphi_{\pi_1,\pi_2\bullet x}^C \circ \widehat{\pi_1}(\varphi_{\pi_2,x}^C)$. Consider the following diagram. As usual, we have written $y = \pi_2 \bullet x$ and $z = \pi_1\pi_2 \bullet x$.

$$\begin{tikzcd}
 & & (1) & & \\
\widehat{\pi_1\pi_2}(A_x \multimap_x B_x) \arrow[d, equal] \arrow[rr, "n_{\pi_1\pi_2}"'] \arrow[rrrr, bend left, "\varphi^C_{\pi_1\pi_2,x}"] & & \widehat{\pi_1\pi_2}(A_x) \multimap_z \widehat{\pi_1\pi_2}(B_x) \arrow[rr, "(\varphi^A_{\pi_1\pi_2,x})^{-1} \multimap_z \varphi^B_{\pi_1\pi_2,x}"'] \arrow[dd, equal] & & A_z \multimap_z B_z \\
\widehat{\pi_1}(\widehat{\pi_2}(A_x \multimap_x B_x)) \arrow[ddr, "\widehat{\pi_1}(n_{\pi_2})"] \arrow[ddddrr, bend right, "\widehat{\pi_1}(\varphi^C_{\pi_2,x})"'] & (2) & & (3) & \\
 & & \widehat{\pi_1}(\widehat{\pi_2}(A_x)) \multimap_z \widehat{\pi_1}(\widehat{\pi_2}(B_x)) \arrow[dr, "\widehat{\pi_1}((\varphi^A_{\pi_2,x})^{-1}) \multimap_z \widehat{\pi_1}(\varphi^B_{\pi_2,x})"'] & & \\
 & \widehat{\pi_1}(\widehat{\pi_2}(A_x) \multimap_y \widehat{\pi_2}(B_x)) \arrow[ur, "n_{\pi_1}"] \arrow[ddr, "\widehat{\pi_1}((\varphi^A_{\pi_2,x})^{-1} \multimap_y \varphi^B_{\pi_2,x})"] & & \widehat{\pi_1}(A_y) \multimap_z \widehat{\pi_1}(B_y) \arrow[uuur, "(\varphi^A_{\pi_1,y})^{-1} \multimap_z \varphi^B_{\pi_1,y}"] & \\
 & (1) & (4) & (1) & \\
 & & \widehat{\pi_1}(A_y \multimap_y B_y) \arrow[uur, "n_{\pi_1}"] \arrow[uuuuurr, bend right, "\varphi^C_{\pi_1,y}"'] & &
\end{tikzcd}$$

The parts labelled (1) commute by definition of φ^C. Part (2) commutes by Remark 7.2. Part (3) commutes because φ^A and φ^B belong to nominal families and therefore satisfy property (a). Part (4) commutes by naturality of n_{π_1}. Therefore, the outside commutes, proving the claim.

For (b), assume $\pi\#x$, so that $\pi \bullet x = x$. We must show that $\varphi^C_{\pi,x} = \pi \bullet (-)$. Consider the following diagram.

$$\begin{tikzcd}
 & (1) & & \\
\widehat{\pi}(A_x \multimap_x B_x) \arrow[r, "n_\pi"] \arrow[rrr, bend left=50, "\pi \bullet (-)"] \arrow[rrr, bend right=50, "\varphi^C_{\pi,x}"'] & \widehat{\pi}(A_x) \multimap_x \widehat{\pi}(B_x) \arrow[rr, bend left, "(\pi \bullet (-))^{-1} \multimap_x \pi \bullet (-)"] \arrow[rr, bend right, "(\varphi^A_{\pi,x})^{-1} \multimap_x \varphi^B_{\pi,x}"'] & (2) & A_x \multimap_x B_x \\
 & (3) & &
\end{tikzcd}$$

Part (3) commutes by definition of φ^C. Part (2) commutes because φ^A and φ^B each satisfy (b). To show part (1), it is (by the universal property of n) sufficient to show that part (1) of the diagram below commutes.

$$\begin{tikzcd}
\widehat{\pi}(A_x \multimap_x B_x) \otimes_x \widehat{\pi}(A_x) \arrow[rrrr, "m_\pi^{-1}"] \arrow[drr, "\pi \bullet (-) \otimes_x \pi \bullet (-)"] \arrow[ddr, "\pi \bullet (-) \otimes_x \mathrm{id}"] \arrow[dddd, "n_\pi \otimes_x \mathrm{id}"'] & & (3) & & \widehat{\pi}((A_x \multimap_x B_x) \otimes_x A_x) \arrow[dll, "\pi \bullet (-)"] \arrow[dddd, "\widehat{\pi}(\varepsilon)"] \\
& (2) & (A_x \multimap_x B_x) \otimes_x A_x \arrow[dr, "\varepsilon"] & (6) & \\
& (A_x \multimap_x B_x) \otimes_x \widehat{\pi}(A_x) \arrow[ur, "\mathrm{id} \otimes_x \pi \bullet (-)"] \arrow[dr, "(\pi \bullet (-) \multimap_x \mathrm{id}) \otimes_x \mathrm{id}"'] & (4) & B_x & \\
& (1) & (\widehat{\pi}(A_x) \multimap_x B_x) \otimes_x \widehat{\pi}(A_x) \arrow[ur, "\varepsilon"'] & (5) & \\
(\widehat{\pi}(A_x) \multimap_x \widehat{\pi}(B_x)) \otimes_x \widehat{\pi}(A_x) \arrow[uurr, "(\mathrm{id} \multimap_x \pi \bullet (-)) \otimes_x \mathrm{id}"'] \arrow[rrrr, "\varepsilon"'] & & & & \widehat{\pi}(B_x) \arrow[uul, "\pi \bullet (-)"']
\end{tikzcd}$$

The outer square commutes by definition of the n map, part (2) commutes by composition, part (3) commutes Lemma 6.7, part (4) commutes by dinaturality of ε, part (5) commutes by naturality of ε, and part (6) commutes by naturality of $(\pi \bullet (-))^{-1}$. Therefore, part (1) commutes, as well. ☐