A unification of graded and substructural logics

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Abstract

Type systems which account for resource sensitive computations can generally be split into two styles: First, substructural logics such as Linear Logic which seek to restrict weakening and contraction and reintroduce them in a controlled manner; And second, graded systems which allow weakening and contraction by default, but track the use of variables quantitatively in some algebraic structure – usually a semiring. We present GRASS (**Gra**ded and **s**ub**s**tructural), a type system which incorporates mechanisms from both of these approaches, thus allowing maximally flexible control over variable usage. Furthermore, GRASS allows grades from an arbitrary collection of grade algebras to coexist in the same system, thus allowing different variables to be controlled with respect to different notions of resource within the same program. We develop the categorical semantics of GRASS, and find that, on the level of categorical semantics, it subsumes multiple established systems such as LNL [4], Adjoint Logic [28], and mGL [32].

Keywords: Programming Languages, Substructural Logic, Graded Modal Types, Categorical Semantics

1 Introduction

The treatment of program variables as resources has seen a rise over the last few decades. While classically variables were treated like propositions which could be reused or discarded at will, the resourceful treatment acknowledges that this is not true in many common computing use cases. Treating variables as resources allows programmers to verify interesting correctness properties, like the absence of memory leaks or use-after-free errors.

Two distinct approaches to the resourceful analysis of programs exist in the literature. *Substructural logics* restrict weakening and contraction, [3] prohibiting variables from being duplicated or discarded by default, then selectively re-introducing these capabilities where needed. In Linear Logic [15], this is done via the of-course modality. Benton generalized Linear Logic to Linear-non-Linear Logic [4], a system which features two sub-logics, one where assumptions are intuitionistic and one where assumptions can be either intuitionistic or linear. Pruiksma et al. [28] further generalized this system to Adjoint Logic, allowing any number of logics, each with its own set of permitted structural rules, to coexist in the same system. This allows programmers to specify exactly which structural rules are applicable to which variable in a program, independently of the other variables.

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[3] Exchange can also be restricted to create non-commutative logics [19], but we do not consider this here.

The second approach is to allow structural rules by default, but to annotate assumptions with so called *grades*, usually drawn from a semiring. In this approach, the control over the usage of assumptions stems from the algebraic structure of the grades with graded contraction being modeled by addition and weakening being modeled by an explicit zero element. Graded systems are commonly parametrized by their grade structure and can therefore model a variety of resource notions. Examples of applications are: Garbage collection [10], liveness analysis [26], security [1,12,21,23,24], dataflow analysis [26,31,30], numerical sensitivity tracking [3,11], and reasoning about probabilistic programs [29].

In this work we present Grass, a type system which unifies these two approaches. Our system allows grades drawn from any number of grade algebras to be used simultaneously. For example, we can type a function `read_decrypt`: $(k :^{\mathsf{Hi}} \texttt{Key}) \to (f :^{1} \texttt{FileHandle}) \to \texttt{Text}$, which reads the contents of an encrypted file f using the decryption key k. The grade annotation Hi ensures that the key is used in a secure context, while the grade annotation 1 ensures that the file handle is properly accessed and closed. Our type system further allows control over graded contraction and weakening. For example two file handles used linearly should not be contractible into one which can be used twice.

We develop a categorical semantics for our type system, based on graded linear exponential comonads [17]. Our type system is assembled from a collection of graded logics connected by morphisms, and this design is mirrored in the categorical semantics. We develop a novel notion of morphisms between graded comonads based on morphisms of actegories [sic] [9]. We show that known categorical models for substructural logics give rise to models of Grass in a straightforward way. Thus, on the level of categorical semantics, Grass is capable of expressing both graded and substructural logics, as well as combinations thereof.

2 Single-mode system

Grass is paratemeterized by a collection of so called *modes*, each of which admits different graded structural rules. In this section we explain our novel weakening and contraction rules in the situation when this collection is instantiated to consist of only one mode. In this case, Grass simplifies to be an ordinary graded type theory, nearly identical to the purely graded fragment of mGL [32], except with modified weakening and contraction rules. We will focus on the structural rules here; the full set of typing rules is given in §A of the appendix [16]. Before we begin with our treatment of the structural rules, we review weakening and contraction in existing graded systems, then explain how we control them further using modes. For contraction, the idea is to restrict it to a set of mutually contractible grades, subject to some conditions. For weakening the situation is simpler: It can just be toggled on or off. We end this section with a discussion of morphisms of modes, which capture the notion that one mode has more structure than another.

2.1 Weakening and contraction in existing systems

We review the behavior of graded weakening and contraction in existing systems. These structural rules are present in most graded type systems, either as explicit rules [13,25,26,32], as admissible rules [10,12,23], or even a mix of the two with an explicit weakening rule, but admissible graded contraction [24]. In this section we will consider typing judgments of the form $\rho \odot \Gamma \vdash t : T$ where t is a term, T is a type and Γ a typing context. Furthermore, ρ is a list of grades of the same length as Γ. Each entry of ρ dictates the capabilties with which the variable in the corresponding position in Γ may be used by t. The graded contraction and weakening rules are now

$$\textsc{Cont}\ \frac{\rho, q_1, q_2 \odot \Gamma, x : A, y : A \vdash e : B}{\rho, q_1 + q_2 \odot \Gamma, z : A \vdash [z/x, z/y]e : B} \qquad \textsc{Weak}\ \frac{\rho \odot \Gamma \vdash e : A}{\rho, 0 \odot \Gamma, x : B \vdash e : A}\ .$$

The contraction rule allows sharing a variable across two sub-expressions in a term, but also requires its capabilities to be split between the two expressions. The weakening rule allows the introduction of additional variables into the context, marked by the grade 0. The graded structural rules can be seen as generalizations of the structural rules available using the of-course modality $!A$ in Linear Logic, where they correspond to maps $!A \multimap !A \otimes !A$ and $!A \multimap I$ respectively. The graded structural rules generalize these to $\Box_{q_1+q_2} A \multimap \Box_{q_1} A \otimes \Box_{q_2} A$ and $\Box_0 A \multimap I$, where the type $\Box_q A$ indicates use at grade q. This allows fine grained tracking of the usage of each hypothesis. Depending on the grade algebra being used, the graded structural rules may simply correspond to the ones of linear logic, but by choosing different kinds of semirings we can enforce different styles of capability tracking. For example, if we use the natural numbers to grade our types we can track the precise number of times variables are being used rather than simply one or zero.

2.2 Modes

In this section we formalize a novel structure used to control weakening and contraction in the graded setting, which we call *modes*. The main insight is that contraction can be controlled by specifying a set of mutually contractible grades called an *ideal*. The notion of ideals is borrowed from ring theory, where ideals are a well-known concept (see e.g. Lang [20]).

Definition 2.1 A *semiring* is a set R together with binary operations $(+), (\cdot)\colon R \times R \to R$ and elements $0, 1 \in R$ such that $(R, +, 0)$ is a commutative monoid, $(R, \cdot, 1)$ is a monoid and such that for all $x, y, z \in R$ we have $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ and $0 \cdot x = 0 = x \cdot 0$. A *preordered semiring* is a semiring equipped with a preorder $(\leq)$ such that $(+)$ and $(\cdot)$ are monotone with respect to $(\leq)$, i.e. whenever $x \leq x'$ and $y \leq y'$, then $x + y \leq x' + y'$ and likewise for $(\cdot)$. Preordered semirings are also called *grade algebras* and their elements are called *grades*.

Example 2.2 (i) The set of natural numbers $\mathbb{N}$ with the usual arithmetic operations forms a semiring. There are several natural choices of preorder on $\mathbb{N}$: The usual ordering $0 \leq 1 \leq \ldots$, its opposite $0 \geq 1 \geq \ldots$, and the *discrete ordering* $n \leq m \iff n = m$.

(ii) The set $\{0, 1\}$ can be equipped with two distinct semiring structures, by either defining $1 + 1 = 0$ or $1 + 1 = 1$. Only the latter structure will be of interest to us. There are three interesting preorders, defined by either $0 \leq 1$ or $0 \geq 1$, or the discrete ordering.

(iii) The *none-one-tons* semiring is the set $\{0, 1, \omega\}$. Addition and multiplication are fixed by the semiring axioms and the equations $1 + 1 = 1 + \omega = \omega + \omega = \omega \cdot \omega = \omega$. Again, there are multiple possible preorders, for example, the preorder generated by $0 \leq \omega$ or the one generated by $0 \leq \omega \geq 1$.

(iv) There is a unique semiring with $0 = 1$, which we denote $\top$.

Definition 2.3 Let R be a semiring. A *(two sided) ideal* in R is a subset $I \subseteq R$ such that $0 \in I$, and whenever $x, y \in I$, then $x + y \in I$, and for any $x \in I$ and $r \in R$, we have $r \cdot x \in I$ and $x \cdot r \in I$.

Example 2.4 (i) Any semiring R admits the trivial ideals $\{0\} \subseteq R$ and $R \subseteq R$.

(ii) The sets $\{0, \omega\} \subseteq \{0, 1, \omega\}$ and $\mathbb{N} \setminus \{1\} \subseteq \mathbb{N}$ are ideals.

(iii) Given a family of ideals I_j, the intersection $\bigcap_j I_j$ is again an ideal. Hence, given any subset $X \subseteq R$ there exists a smallest ideal containing X: There exists an ideal containing X since $R \subseteq R$ is an ideal itself. Now consider the set of all such ideals and take their intersection.

Definition 2.5 A mode m is a triple $(R_\mathsf{m}, \mathsf{Cont}(\mathsf{m}), \mathsf{Weak}(\mathsf{m}))$ where R_m is a grade algebra, $\mathsf{Cont}(\mathsf{m}) \subseteq R_\mathsf{m}$ is an ideal and $\mathsf{Weak}(\mathsf{m}) \in \{\mathsf{true}, \mathsf{false}\}$.

Remark 2.6 Our notion of modes is a generalization of the notion of modes given in Adjoint Logic [28]. There, a mode is equipped with two boolean values, indicating whether weakening and contraction are permitted. In the graded setting, we can control weakening and contraction more granularly by enabling them only for a subset of grades. Contraction is modeled by a set of mutually contractible grades. We will see below how the conditions that this set is an ideal arise naturally in the syntax. While it may appear that weakening is still controlled by a boolean, the situation is a bit more subtle: Most graded systems, including GRASS, have a subsumption rule, which states that grades need only be upper bounds on the actual usage of variables. Thus, when weakening is enabled, unused variables can be introduced at any grade $q \geq 0$, by applying weakening followed by subsumption. We do not require 0 to be the bottom element of a grade algebra, so weakening is controlled by the predicate $\mathsf{Weak}(\mathsf{m})$ and the choice of preorder.

Example 2.7 The following modes will serve as running examples: $\mathsf{U} = (\top, \top, \mathsf{true})$, $\mathsf{R} = (\top, \top, \mathsf{false})$, $\mathsf{A} = (\{0, 1\}, \{0\}, \mathsf{true})$ and $\mathsf{L} = (\mathbb{N}, \{0\}, \mathsf{false})$. We will see in Example 2.8 how these modes can recover intuitionistic, relevant, affine and linear logic respectively.

2.3 Restricted structural rules

We can now describe the controlled structural of Grass. Fix a mode $\mathsf{m} = (R_\mathsf{m}, \mathsf{Cont}(\mathsf{m}), \mathsf{Weak}(\mathsf{m}))$. The new rules for contraction and weakening are as follows:

$$\text{S.Cont}\ \frac{q_1, q_2 \in \mathsf{Cont}(\mathsf{m}) \quad \rho, q_1, q_2 \odot \Gamma, x : A, y : A \vdash e : B}{\rho, q_1 + q_2 \odot \Gamma, z : A \vdash [z/x, z/y]e : B} \qquad \text{S.Weak}\ \frac{\mathsf{Weak}(\mathsf{m}) \quad \rho \odot \Gamma \vdash e : A}{\rho, 0 \odot \Gamma, x : B \vdash e : A}\ .$$

The weakening rule is straightforwardly modified: It can be toggled on or off by the mode. Contraction is only allowed when both grades are elements of $\mathsf{Cont}(\mathsf{m})$. The requirement that $\mathsf{Cont}(\mathsf{m})$ is an ideal arises naturally as follows. First, consider a judgment $\rho, q_1, q_2, q_3 \odot \Gamma, x_1 : A, x_2 : A, x_3 : A \vdash t : B$ where $q_1, q_2, q_3 \in \mathsf{Cont}(\mathsf{m})$. We want $\mathsf{Cont}(\mathsf{m})$ to act as a set of mutually contractible grades, so it should be possible to contract this into a term $\rho, q_1 + q_2 + q_3 \odot \Gamma, z : A \vdash [z/x_1, z/x_2, z/x_3]t : B$. Since the contraction rule only allows contracting two variables at a time, we must pick an order: We either contract x_1 with x_2 and then the result with x_3, or x_2 and x_3 first and then the result with x_1. The condition that $\mathsf{Cont}(\mathsf{m})$ is closed under addition means that both orderings are always possible, and in fact the resulting derivations should be considered equal. Similarly, the condition that $0 \in \mathsf{Cont}(\mathsf{m})$ ensures that contraction interacts well with weakening in the sense that for $q \in \mathsf{Cont}(\mathsf{m})$ we have the derivation

$$\cfrac{\cfrac{\rho, q \odot \Gamma, x : A \vdash e : B}{\rho, q, 0 \odot \Gamma, x : A, y : A \vdash e : B}\ \text{S.Weak}}{\rho, q + 0 \odot \Gamma, z : A \vdash [z/x]e : B}\ \text{S.Cont}$$

and the resulting term should be considered equal to e. Finally, contraction should also interact well with substitution. In graded type systems, substitution takes the form

$$\frac{\rho, r \odot \Gamma, x : A \vdash e : B \qquad \sigma \odot \Delta \vdash t : A}{\rho, r \cdot \sigma \odot \Gamma, \Delta \vdash [t/x]e : B}\ .$$

Given terms $\rho, r \odot \Gamma, x : A \vdash e : B$ and $\sigma, q_1, q_2 \odot \Delta, z_1 : T, z_2 : T \vdash t : A$ with $q_1, q_2 \in \mathsf{Cont}(\mathsf{m})$ we can apply contraction, obtaining $\sigma, q_1 + q_2 \odot \Delta, z : T \vdash [z/z_1, z/z_2]t : A$ then perform substitution. The condition that $\mathsf{Cont}(\mathsf{m})$ is an ideal ensures that the other ordering is possible too: If we substitute first, obtaining $\rho, r \cdot \sigma, r \cdot q_1, r \cdot q_2 \odot \Gamma, \Delta, z_1 : T, z_2 : T \vdash [t/x]e : B$ then we may still apply contraction. Note that both orders of these operations result in the same term.

Example 2.8 We can recover non-graded logics by using appropriate modes.

(i) Using the mode $\mathsf{U} = (\top, \top, \mathsf{true})$ we recover intuitionistic logic: Contraction and weakening are permitted.

(ii) Using mode $\mathsf{R} = (\top, \top, \mathsf{false})$ recovers *relevant logic*: There is only one grade that may be contracted with itself, so variables may be reused arbitrarily. However, weakening is disabled, so unused variables are not allowed.

(iii) Using the mode $\mathsf{A} = (\{0, 1\}, \{0\}, \mathsf{true})$ with ordering $0 \leq 1$, we recover *affine logic*. Variables graded 1 may be unused by applying a combination of the weakening and subsumption rules. Contraction is only allowed for unused variables graded 0, so reuse is impossible.

(iv) Using the mode $(\{0, 1\}, \{0\}, \mathsf{false})$ we almost recover linear logic. Variables graded 1 are guaranteed to have linear usage. The system we obtain here is not exactly linear logic (hence "almost"), since variables graded 0 can still be introduced explicitly using the graded modality $\Box_0 A$, or by applying the elimination rule for the unit type. (The unit type has no computational content so its terms may be eliminated at any grade.)

(v) Similarly, grading by the mode $\mathsf{L} = (\mathbb{N}, \{0\}, \mathsf{false})$ with discrete ordering tracks the number of free occurences of variables in a term exactly. Since contraction and weakening are disabled, variables graded by $n \neq 1$ must be introduced explicitly via the graded modality $\Box_n A$, or the unit type's elimination rule. Thus, we recover another variant on linear logic.

Example 2.9 Using the mode $(R = \{0, 1, \omega\}, \{0, \omega\}, \mathsf{true})$, the grades mean unused (0), linear (1), and unrestricted (ω). Furthermore, two variables graded 1 may not be contracted into one variable graded $1 + 1 = \omega$. This is useful, for example, when dealing with file handles: Two file handles used linearly cannot be replaced by one file handle with unrestricted use. Similar considerations apply to the mode $(\mathbb{N}, \mathbb{N} \setminus \{1\}, \mathsf{true})$: Contraction in only allowed for variables which are known to be used non-linearly.

2.4 Comparing modes

Definition 2.10 A morphism of semirings $R \to S$ is a function $f \colon R \to S$ such that $f(0) = 0$, $f(1) = 1$, and for all $x, y \in R$, $f(x + y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$. A morphism of grade algebras $f \colon R \to S$ is a morphism of semirings which is also monotone, i.e. $x \le y$ implies $f(x) \le f(y)$. A morphism of modes $\mathsf{m} \to \mathsf{n}$ is a morphism $f \colon R_\mathsf{m} \to R_\mathsf{n}$ of the underlying grade algebras such that for each $x \in \mathsf{Cont}(\mathsf{m})$, we have $f(x) \in \mathsf{Cont}(\mathsf{n})$ and such that the boolean formula $\mathsf{Weak}(\mathsf{m}) \to \mathsf{Weak}(\mathsf{n})$ is true. This definition yields a category of modes, denoted **Mode**.

A morphism of modes $\varphi \colon \mathsf{m} \to \mathsf{n}$ implies that n has "more structure" than m in an appropriate way. We give some intuition for this: Given a derivable judgment $\rho \odot \Gamma \vdash t : A$ under mode m there exists a corresponding judgment $\varphi(\rho) \odot \Gamma \vdash t : A$ which is derivable under mode n. Here, $\varphi(\rho)$ is the result of applying φ to each element of ρ. In other words, there is a translation function. This translation is not quite the identity on types and terms, since some types and terms feature grade annotations, which also need to be transported along φ. The proof that this translation is sound is straightforward by induction, since morphisms of modes preserve all structure carried by a mode. In particular, each application of the weakening or contraction rules at mode m can be mirrored at n, which we take to mean that n has more structure than m. This intuition has two interesing special cases: The category **Mode** has a terminal object U and an initial object L. We already saw in Example 2.8 that these modes recover intuitionistic and linear logic respectively. Therefore it makes sense to think of the terminal and initial modes as having maximal and minimal structure respectively.

3 Syntax of Grass

In this section we explain the syntax and typing rules of Grass. Our system is parametrized by a family of modes, with each type belonging to one mode. Like with Adjoint Logic [28], the modes are arranged in a preorder, with modes higher in the order admitting more structural rules than those lower in the order, as discussed in §2.4. First, we present several definitions used throughout the paper.

Definition 3.1 Grass is parametrized by a functor $S \to \mathbf{Mode}$, where S is some preordered set. We usually elide mention of this functor, instead thinking of its image as a preordered set of modes $\mathsf{m}, \mathsf{n}, \mathsf{l}, \dots$ together with a coherent system of morphisms $\varphi_{\mathsf{m},\mathsf{n}} \colon \mathsf{m} \to \mathsf{n}$, whenever $\mathsf{m} \le \mathsf{n}$. We fix such data for the remainder of this text.

Definition 3.2 Grades are denoted by the letters q, r, s. They may be drawn from any of the grade algebras R_m. *Grade vectors* are finite lists of grades, denoted by the letters ρ, σ, τ. Grade vectors are *heterogenous*, i.e. entries of one grade vector may be drawn from more than one grade algebra.

Definition 3.3 Let $\mathsf{m} \le \mathsf{n}$ be modes and let $q \in R_\mathsf{m}$ and $r \in R_\mathsf{n}$. We define $qr = \varphi_{\mathsf{m},\mathsf{n}}(q) \cdot r \in R_\mathsf{n}$. We extend this notation to scalar multiplication: If $q \in R_\mathsf{m}$ and $\rho = (r_1, \dots, r_k)$ with $r_i \in R_{\mathsf{m}_i}$ and $\mathsf{m}_i \ge \mathsf{m}$ for $i \in \{1, \dots, k\}$, then we write $q\rho = (qr_1, \dots, qr_k) = (\varphi_{\mathsf{m},\mathsf{m}_1}(q) \cdot r_1, \dots, \varphi_{\mathsf{m},\mathsf{m}_k}(q) \cdot r_k)$.

Definition 3.4 Types are denoted A, B, C, T. Each type in Grass belongs to a unique mode and we write $A \in \mathsf{Type}(\mathsf{m})$ to indicate that the type A is well-formed and belongs to mode m. In this case, variables of type A are graded by grades drawn from R_m, and structural rules may be applied to them as prescribed by $\mathsf{Weak}(\mathsf{m})$ and $\mathsf{Cont}(\mathsf{m})$. The rules for well-formed types are given in Figure 1.

$$\textsc{Ty.Unit}\ \frac{}{\mathbf{I}_{\mathsf{m}} \in \mathsf{Type}(\mathsf{m})} \qquad \textsc{Ty.Pair}\ \frac{A \in \mathsf{Type}(\mathsf{m}) \qquad B \in \mathsf{Type}(\mathsf{m})}{A \otimes B \in \mathsf{Type}(\mathsf{m})}$$

$$\textsc{Ty.Fun}\ \frac{\mathsf{n} \leq \mathsf{m} \qquad q \in R_{\mathsf{m}} \qquad A \in \mathsf{Type}(\mathsf{m}) \qquad B \in \mathsf{Type}(\mathsf{n})}{A^{q:\mathsf{m}} \multimap B \in \mathsf{Type}(\mathsf{n})} \qquad \textsc{Ty.Sum}\ \frac{A \in \mathsf{Type}(\mathsf{m}) \qquad B \in \mathsf{Type}(\mathsf{m})}{A \oplus B \in \mathsf{Type}(\mathsf{m})}$$

$$\textsc{Ty.Drop}\ \frac{\mathsf{n} \leq \mathsf{m} \qquad q \in R_{\mathsf{m}} \qquad A \in \mathsf{Type}(\mathsf{m})}{\downarrow^{q}_{\mathsf{n}\leq\mathsf{m}} A \in \mathsf{Type}(\mathsf{n})} \qquad \textsc{Ty.Raise}\ \frac{\mathsf{m} \leq \mathsf{n} \qquad A \in \mathsf{Type}(\mathsf{m})}{\uparrow_{\mathsf{m}\leq\mathsf{n}} A \in \mathsf{Type}(\mathsf{n})}$$

Fig. 1. GRASS rules for well-formed types

$$\textsc{Var}\ \frac{A \in \mathsf{Type}(\mathsf{m})}{1 \mid \mathsf{m} \odot x : A \vdash_{\mathsf{m}} x \colon A} \qquad \textsc{Weak}\ \frac{\mathsf{Weak}(\mathsf{m}) \qquad \mathsf{n} \leq \mathsf{m} \qquad B \in \mathsf{Type}(\mathsf{m}) \qquad \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} e \colon A}{\rho, 0 \mid \mathsf{M}, \mathsf{m} \odot \Gamma, x : B \vdash_{\mathsf{n}} e \colon A}$$

$$\textsc{Cont}\ \frac{q_1, q_2 \in \mathsf{Cont}(\mathsf{m}) \qquad \rho, q_1, q_2 \mid \mathsf{M}, \mathsf{m}, \mathsf{m} \odot \Gamma, x : A, y : A \vdash_{\mathsf{n}} e \colon B}{\rho, q_1 + q_2 \mid \mathsf{M}, \mathsf{m} \odot \Gamma, z : A \vdash_{\mathsf{n}} [z/x, z/y]e \colon B}$$

$$\textsc{Sub}\ \frac{\rho \leq \sigma \qquad \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} e \colon A}{\sigma \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} e \colon A} \qquad \textsc{Unit.i}\ \frac{}{\emptyset \mid \emptyset \odot \emptyset \vdash_{\mathsf{m}} \star_{\mathsf{m}} \colon \mathbf{I}_{\mathsf{m}}}$$

$$\textsc{Unit.e}\ \frac{\sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{m}} e \colon \mathbf{I}_{\mathsf{m}} \qquad \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \colon A \qquad q \in R_{\mathsf{m}}}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{m}} \mathsf{let}_{@q} \star_{\mathsf{m}} = e \;\mathsf{in}\; t \colon A}$$

$$\textsc{Fun.i}\ \frac{\rho, q \mid \mathsf{M}, \mathsf{m} \odot \Gamma, x : A \vdash_{\mathsf{n}} t \colon B}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} \lambda x.t \colon A^{q:\mathsf{m}} \multimap B} \qquad \textsc{Fun.e}\ \frac{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} t \colon A^{q:\mathsf{m}} \multimap B \qquad \sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{m}} e \colon A}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{n}} t\, e \colon B}$$

$$\textsc{Pair.i}\ \frac{\rho \mid \mathsf{M}_1 \odot \Gamma_1 \vdash_{\mathsf{m}} t_1 \colon A_1 \qquad \sigma \mid \mathsf{M}_2 \odot \Gamma_2 \vdash_{\mathsf{m}} t_2 \colon A_2}{\rho, \sigma \mid \mathsf{M}_1, \mathsf{M}_2 \odot \Gamma_1, \Gamma_2 \vdash_{\mathsf{m}} (t_1, t_2) \colon A_1 \otimes A_2}$$

$$\textsc{Pair.e}\ \frac{\sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{m}} e \colon A_1 \otimes A_2 \qquad \rho, q, q \mid \mathsf{M}, \mathsf{m}, \mathsf{m} \odot \Gamma, x_1 : A_1, x_2 : A_2 \vdash_{\mathsf{n}} t \colon B}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{n}} \mathsf{let}_{@q} (x_1, x_2) = e \;\mathsf{in}\; t \colon B}$$

$$\textsc{Sum.il}\ \frac{B \in \mathsf{Type}(\mathsf{m}) \qquad \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} e \colon A}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} \mathsf{inl}\; e \colon A \oplus B} \qquad \textsc{Sum.ir}\ \frac{A \in \mathsf{Type}(\mathsf{m}) \qquad \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} e \colon B}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} \mathsf{inr}\; e \colon A \oplus B}$$

$$\textsc{Sum.e}\ \frac{q \geq 1 \qquad \sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{n}} e \colon A_1 \oplus A_2 \qquad \rho, q \mid \mathsf{M}, \mathsf{n} \odot \Gamma, x_1 : A_1 \vdash_{\mathsf{m}} t_1 \colon B \qquad \rho, q \mid \mathsf{M}, \mathsf{n} \odot \Gamma, x_2 : A_2 \vdash_{\mathsf{m}} t_2 \colon B}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{m}} \mathsf{case}_q(e; x_1.t_1; x_2.t_2) \colon B}$$

$$\textsc{Drop.i}\ \frac{\mathsf{n} \leq \mathsf{m} \qquad q \in R_{\mathsf{m}} \qquad \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} e \colon A}{q\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} \downarrow^{q}_{\mathsf{n}\leq\mathsf{m}} e \colon \downarrow^{q}_{\mathsf{n}\leq\mathsf{m}} A}$$

$$\textsc{Drop.e}\ \frac{\mathsf{l} \leq \mathsf{n} \leq \mathsf{m} \qquad \sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{n}} e \colon \downarrow^{q}_{\mathsf{n}\leq\mathsf{m}} A \qquad \rho, q \mid \mathsf{M}, \mathsf{m} \odot \Gamma, x : A \vdash_{\mathsf{l}} t \colon B}{\rho, \sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{l}} \mathsf{let}_{@q} \downarrow_{\mathsf{n}\leq\mathsf{m}} x = e \;\mathsf{in}\; t \colon B}$$

$$\textsc{Raise.i}\ \frac{\mathsf{m} \leq \mathsf{n} \leq \mathsf{M} \qquad \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} e \colon A}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} \uparrow_{\mathsf{m}\leq\mathsf{n}} e \colon \uparrow_{\mathsf{m}\leq\mathsf{n}} A} \qquad \textsc{Raise.e}\ \frac{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} e \colon \uparrow_{\mathsf{m}\leq\mathsf{n}} A}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} \uparrow^{-1}_{\mathsf{m}\leq\mathsf{n}} e \colon A}$$

Fig. 2. GRASS typing rules

Definition 3.5 Typing judgments in GRASS have the form $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \colon T$. The roles of Γ, t, and T are as before. There are two new components in the judgment: The annotation m on the turnstile indicates that $T \in \mathsf{Type}(\mathsf{m})$, and we say that this typing judgment is *made at mode* m. The other new component is M, which is a list of modes, also called a *mode vector*. If $\rho = (r_1, \ldots, r_k)$ and $\mathsf{M} = (\mathsf{m}_1, \ldots, \mathsf{m}_k)$ and $\Gamma = x_1 : A_1, \ldots, x_k : A_k$, then the above judgment indicates that $A_i \in \mathsf{Type}(\mathsf{m}_i)$ and that the variable x_i is used with capabilities indicated by r_i.

Definition 3.6 Typing contexts are denoted by Γ, Δ. When Γ and Δ are contexts, their concatenation is written Γ, Δ and is only well-defined when the contexts Γ and Δ have no variables in common. Whenever we use context concatenation, we implicitly assume that this condition is satisfied.

In any derivable typing judgment $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \colon T$, each mode n occuring in M must satisfy $\mathsf{n} \geq \mathsf{m}$, which we write concisely as $\mathsf{M} \geq \mathsf{m}$. A similar requirement is present in Linear-non-Linear Logic [4], where

$$\textsc{Unit}.\beta\quad \frac{\begin{array}{c}\sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{m}} \star_{\mathsf{m}} \colon \mathbf{I}_{\mathsf{m}} \\ \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} e \colon A\end{array}}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{m}} \mathsf{let}_{@q} \star_{\mathsf{m}} = \star_{\mathsf{m}} \;\mathsf{in}\; e \equiv_\beta e \colon A}$$

$$\textsc{Pair}.\beta\quad \frac{\begin{array}{c}\sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{m}} (t_1, t_2) \colon A_1 \otimes A_2 \\ \rho, q, q \mid \mathsf{M}, \mathsf{m}, \mathsf{m} \odot \Gamma, x_1 : A_1, x_2 : A_2 \vdash_{\mathsf{n}} e \colon A\end{array}}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{m}} \mathsf{let}_{@q} (x_1, x_2) = (t_1, t_2) \;\mathsf{in}\; e \equiv_\beta [t_1/x_1, t_2/x_2] e \colon A}$$

$$\textsc{Fun}.\beta\quad \frac{\begin{array}{c}\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} \lambda x.e \colon B^{q:\mathsf{m}} \multimap A \\ \sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{m}} t \colon B\end{array}}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{n}} (\lambda x.e)\, t \equiv_\beta [t/x] e \colon A}$$

$$\textsc{Sum}.\beta.\textsc{L}\quad \frac{\begin{array}{l}q \geq 1 \\ \sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{n}} \mathsf{inl}\; t \colon A_1 \oplus A_2 \\ \rho, q \mid \mathsf{M}, \mathsf{n} \odot \Gamma, x_1 : A_1 \vdash_{\mathsf{m}} e_1 \colon A \\ \rho, q \mid \mathsf{M}, \mathsf{n} \odot \Gamma, x_2 : A_2 \vdash_{\mathsf{m}} e_2 \colon A\end{array}}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{m}} \mathsf{case}_q (\mathsf{inl}\; t; x_1.e_1; x_2.e_2) \equiv_\beta [t/x_1] e_1 \colon A}$$

$$\textsc{Sum}.\beta.\textsc{R}\quad \frac{\begin{array}{l}q \geq 1 \\ \sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{n}} \mathsf{inr}\; t \colon A_1 \oplus A_2 \\ \rho, q \mid \mathsf{M}, \mathsf{n} \odot \Gamma, x_1 : A_1 \vdash_{\mathsf{m}} e_1 \colon A \\ \rho, q \mid \mathsf{M}, \mathsf{n} \odot \Gamma, x_2 : A_2 \vdash_{\mathsf{m}} e_2 \colon A\end{array}}{\rho, q\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{m}} \mathsf{case}_q (\mathsf{inl}\; t; x_1.e_1; x_2.e_2) \equiv_\beta [t/x_2] e_2 \colon A}$$

$$\textsc{Raise}.\beta\quad \frac{\begin{array}{l}\mathsf{m} \leq \mathsf{n} \\ \rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} \uparrow_{\mathsf{m} \leq \mathsf{n}} t \colon \uparrow_{\mathsf{m} \leq \mathsf{n}} A\end{array}}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} \uparrow^{-1}_{\mathsf{m} \leq \mathsf{n}} (\uparrow_{\mathsf{m} \leq \mathsf{n}} t) \equiv_\beta t \colon A}$$

$$\textsc{Drop}.\beta\quad \frac{\begin{array}{l}\mathsf{l} \leq \mathsf{n} \leq \mathsf{m} \\ \sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{n}} \downarrow^q_{\mathsf{n} \leq \mathsf{m}} t \colon \downarrow^q_{\mathsf{n} \leq \mathsf{m}} A \\ \rho, q \mid \mathsf{M}, \mathsf{m} \odot \Gamma, x : A \vdash_{\mathsf{l}} e \colon B\end{array}}{\rho, \sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{l}} \mathsf{let}_{@q} \downarrow_{\mathsf{n} \leq \mathsf{m}} x = \downarrow^q_{\mathsf{n} \leq \mathsf{m}} t \;\mathsf{in}\; e \equiv_\beta [t/x] e \colon B}$$

$$\textsc{Unit}.\eta\quad \frac{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \colon \mathbf{I}_{\mathsf{m}}}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \equiv_\eta \mathsf{let}_{@1} \star_{\mathsf{m}} = t \;\mathsf{in}\; \star_{\mathsf{m}} \colon \mathbf{I}_{\mathsf{m}}}$$

$$\textsc{Pair}.\eta\quad \frac{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \colon A_1 \otimes A_2}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \equiv_\eta \mathsf{let}_{@1} (x_1, x_2) = t \;\mathsf{in}\; (x_1, x_2) \colon A_1 \otimes A_2}$$

$$\textsc{Fun}.\eta\quad \frac{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} t \colon A^{q:\mathsf{m}} \multimap B}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \equiv_\eta \lambda x.(t\, x) \colon A^{q:\mathsf{m}} \multimap B}$$

$$\textsc{Sum}.\eta\quad \frac{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \colon A_1 \oplus A_2}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \equiv_\eta \mathsf{case}_1 (t; x_1.\,\mathsf{inl}\; x_1; x_2.\,\mathsf{inr}\; x_2) \colon A_1 \oplus A_2}$$

$$\textsc{Raise}.\eta\quad \frac{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \colon \uparrow_{\mathsf{n} \leq \mathsf{m}} A}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \equiv_\eta \uparrow_{\mathsf{n} \leq \mathsf{m}} (\uparrow^{-1}_{\mathsf{n} \leq \mathsf{m}} t) \colon \uparrow_{\mathsf{n} \leq \mathsf{m}} A}$$

$$\textsc{Drop}.\eta\quad \frac{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} t \colon \downarrow^q_{\mathsf{n} \leq \mathsf{m}} A}{\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{n}} t \equiv_\eta \mathsf{let}_{@q} \downarrow_{\mathsf{n} \leq \mathsf{m}} x = t \;\mathsf{in} \downarrow^q_{\mathsf{n} \leq \mathsf{m}} x \colon \downarrow^q_{\mathsf{n} \leq \mathsf{m}} A}$$

Fig. 3. Grass $\beta\eta$-conversions

linear terms may depend on non-linear variables, but not vice-versa, and was also observed by Pruiksma et al. [28], where it is called *independence*. The typing rules of Grass are set up to ensure that independence holds in all derivable judgments. In Grass, independence also ensures that all scalar multiplications occuring in the typing rules are well-defined (cf. Definition 3.3). We discuss the typing rules in detail in the following paragraphs.

We begin with the structural rules. The grade 1 corresponds to linear usage, and is thus used in the variable rule (Var). The rules for weakening and contraction are as before, only modified to account for the presence of different modes. Contraction (rule Cont) is only allowed between grades of the same mode. In the weakening rule (Weak) the condition $\mathsf{n} \leq \mathsf{m}$ ensures that independence holds in the conclusion, if it holds in the assumption. We assume an implicit exchange rule, which allows reordering of contexts. When this occurs, the grade and mode vectors must be reordered accordingly.

The type $\downarrow^q_{\mathsf{n} \leq \mathsf{m}} A$ is defined for $A \in \mathsf{Type}(\mathsf{m})$, $q \in R_{\mathsf{m}}$ and $\mathsf{n} \leq \mathsf{m}$ (rule Ty.Drop). Terms of this type behave like terms of type A, with the added constraint that they must be used at grade q. Furthermore, the type $\downarrow^q_{\mathsf{n} \leq \mathsf{m}} A$ belongs to mode n and is therefore subject to the structural rules imposed by n. The introduction rule (Drop.i) promotes a term t of type A to a term $\downarrow^q_{\mathsf{n} \leq \mathsf{m}} t$ of type $\downarrow^q_{\mathsf{n} \leq \mathsf{m}} A$. The resources used to produce t are multiplied by q during this promotion, reflecting the q-fold reuse later. The elimination rule (Drop.e) uses a let binding $\mathsf{let}_{@q} \downarrow_{\mathsf{n} \leq \mathsf{m}} x = e \;\mathsf{in}\; t$ to extract a term of type A from e, and make it available to the body t, which then must use that term at grade q. This type combines two distinct feature sets from existing literature: First, some graded type theories (e.g. [1,8,10,13,24]) include a graded modality, commonly written $\Box_q A$. We can recover such a type in Grass by setting $\Box_q A := \downarrow^q_{\mathsf{m}\leq\mathsf{m}} A$. Second, $\downarrow^q_{\mathsf{n}\leq\mathsf{m}} A$ is an analogue of $\mathsf{F}A$ from LNL [4], and $\mathsf{Grd}_q A$ from mGL [32], since these constructs move types to a mode with fewer structural rules. Our notation mirrors that of Pruiksma et al. [28].

The type $\uparrow_{\mathsf{n}\leq\mathsf{m}} A$ is defined for $A \in \mathsf{Type}(\mathsf{n})$ and $\mathsf{n} \leq \mathsf{m}$ (rule Ty.Raise). The introduction rule (Raise.i) constructs a term $\uparrow_{\mathsf{n}\leq\mathsf{m}} e : \uparrow_{\mathsf{n}\leq\mathsf{m}} A$ from a term $e : A$. The produced term and its type belong to mode m and thus have fewer structural restrictions imposed upon them. The elimination rule (Raise.e) allows us to extract a term of type A from a term of type $\uparrow_{\mathsf{n}\leq\mathsf{m}} A$ directly. The type $\uparrow_{\mathsf{n}\leq\mathsf{m}} A$ is the analogue of $\mathsf{G}A$ from LNL and Lin from mGL.

Grass includes a subsumption rule Sub. This rule means that a grade vector ρ in a judgment $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t \colon A$ need only be an upper bound on the actual usage of the variables in Γ within the term e. The relation $\rho \leq \sigma$ is defined inductively by

$$\frac{}{\emptyset \leq \emptyset} \qquad\qquad \frac{\rho \leq \sigma \qquad r \in R_{\mathsf{m}} \qquad q \in R_{\mathsf{m}} \qquad r \leq q}{(\rho, r) \leq (\sigma, q)}.$$

When the preorder is discrete, this rule is effectively disabled.

For each mode m, we have a unit type $\mathbf{I}_{\mathsf{m}}$ (rule Ty.Unit) which admits a closed term $\star_{\mathsf{m}}$ that can be produced without using any resources (rule Unit.i). Terms of type $\mathbf{I}_{\mathsf{m}}$ hold no computational information, and may thus be eliminated (rule Unit.e) at any grade q by the pattern matching construct $\mathsf{let}_{@q} \star_{\mathsf{m}} = e \,\mathsf{in}\, t$.

The tensor product type $A_1 \otimes A_2$ is defined only when the types A_1 and A_2 belong to the same mode. Its introduction rule (Pair.i) is standard, allowing the construction of a pair (t_1, t_2) from terms t_1 and t_2. The pair elimination rule (Pair.e) functions by pattern matching. It asserts that using a pair at grade q is the same thing as using each of its components separately at grade q. Since term reuse is modeled by multiplication, the grades used to produce the pair are multiplied by q as part of the elimination rule.

The function type $A^{q:\mathsf{m}} \multimap B$ is defined when $A \in \mathsf{Type}(\mathsf{m})$ and $B \in \mathsf{Type}(\mathsf{n})$ with $\mathsf{m} \geq \mathsf{n}$. The requirement that $\mathsf{m} \geq \mathsf{n}$ is an instance of independence. The grade annotation q indicates how functions of this type use their parameters. Functions are introduced (rule Fun.i) using λ-abstraction. The grade annotation q corresponds to the the use of the abstacted variable in the function body. Elimination (Fun.e) is done by function application. Since grade multiplication corresponds to reuse, the grades used to produce the parameter e are multiplied by q during application.

Like the product, the sum type $A_1 \oplus A_2$ is defined between types of the same mode. Its introduction forms (Sum.il, Sum.ir) are standard, injecting a term e of type A_1 (resp. A_2) into the sum. These injections do not affect the grades used to produce e. Elimination occurs by cases: to use a term of type $A_1 \oplus A_2$ at grade q, we must specify two branches, which use a term of A_1 (resp. A_2) at grade q.

Example 3.7 Let $\mathsf{m}_1, \mathsf{m}_2$ be two modes, and let L be the initial mode as discussed in §2.4. Instantiate Grass with modes $\mathsf{m}_1 > \mathsf{L} < \mathsf{m}_2$. Write $\Box^q_{\mathsf{m}} A = \downarrow^q_{\mathsf{L}\leq\mathsf{m}}(\uparrow_{\mathsf{L}\leq\mathsf{m}} A)$. Then L behaves like linear logic, with two graded modalities $\Box^q_{\mathsf{m}_1} A$ and $\Box^q_{\mathsf{m}_2} A$ which can account for two distinct notions of resource usage. This is a generalization of mGL allowing grades from two different grade algebras.

Example 3.8 Using the modes $\mathsf{L} < \mathsf{U}$ of Example 2.8, we obtain LNL [4]. If we also use the modes A and R, we obtain a combined system in the style of Adjoint Logic [28].

Theorem 3.9 (Substitution) *If* $\rho, r \mid \mathsf{M}, \mathsf{m} \odot \Gamma, x : A \vdash_{\mathsf{n}} e \colon B$ *and* $\sigma \mid \mathsf{N} \odot \Delta \vdash_{\mathsf{m}} t \colon A$*, then*

$$\rho, r\sigma \mid \mathsf{M}, \mathsf{N} \odot \Gamma, \Delta \vdash_{\mathsf{n}} [t/x]e \colon B$$

Proof. See §B.1 in the appendix [16]. ◻

We include a set of β- and η-conversion rules which are given in Figure 3. The β-rules are mostly standard: Pattern matching eliminators reduce when the scrutinee of the match is a constructor for the respective type. In the pattern matching form for the unit type, the grade annotation q gets ignored since the term $\star_{\mathsf{m}}$ carries no computational information. Function applications reduce by substitution into the function body, when the function in question is a lambda. The type $\uparrow_{\mathsf{m}\leq\mathsf{n}} A$ is a negative type with an explicit destructor $\uparrow^{-1}_{\mathsf{m}\le\mathsf{n}}\, e$, rather than a pattern matching one, which acts as a left inverse to the constructor $\uparrow_{\mathsf{m}\le\mathsf{n}}\, e$. The η-rules are similarly standard: eliminating then reconstructing a term recovers that term. These conversions preserve typing:

Theorem 3.10 (Type preservation) *The β- and η-conversions preserve typing and grading, i.e.*

(i) *Whenever $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t_1 \equiv_\beta t_2 \colon A$, then $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t_1 \colon A$, and $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t_2 \colon A$.*

(ii) *Whenever $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t_1 \equiv_\eta t_2 \colon A$, then $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t_1 \colon A$, and $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t_2 \colon A$.*

Proof. See §§B.2, B.3 in the appendix [16]. □

4 Categorical semantics

We present our categorical semantics. The structure of the categorical models mirrors that of the syntax: Grass is composed out of several graded logics, one for each mode, tied together by a system of modalities $\downarrow^{q}_{\mathsf{n}\le\mathsf{m}} A$ and $\uparrow_{\mathsf{m}\le\mathsf{n}} A$ mediating between them. Similiarly, a model for Grass consists of several models for ordinary graded logic, tied together by an appropriate system of morphisms, here realized as adjunctions with some additional structure. To appropriately capture the modified weakening and contraction rules, we must make some changes relative to the existing literature. Because of this, we set up all necessary terminology from scratch. The categorical semantics presented here do not include the sum type; Adding the necessary structure to model it is straightforward. The following definitions are adapted from Katsumata [17].

Definition 4.1 Let $\mathsf{m} = (R_{\mathsf{m}}, \mathsf{Cont}(\mathsf{m}), \mathsf{Weak}(\mathsf{m}))$ be a mode. The preorder on R_{m} equips the sets R_{m} and $\mathsf{Cont}(\mathsf{m})$ with categorical structures. We write $(R_{\mathsf{m}}, \cdot, 1)$ for the symmetric monoidal structure on R_{m} whose tensor product is multiplication and tensor unit is $1 \in R_{\mathsf{m}}$. Similarly, we write $(\mathsf{Cont}(\mathsf{m}), +, 0)$ for the symmetric monoidal structure on $\mathsf{Cont}(\mathsf{m})$ whose tensor product is addition and tensor unit is $0 \in \mathsf{Cont}(\mathsf{m})$.

Definition 4.2 Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. Write $\mathbf{SMC}(\mathcal{C}, \mathcal{C})$ for the category of strong symmetric monoidal endofunctors on $\mathcal{C}$ and monoidal natural transformations between them. This category can be equipped with a monoidal structure in two ways:

(i) Pointwise: For $F, G \in \mathbf{SMC}(\mathcal{C}, \mathcal{C})$, their tensor product is $(F \mathbin{\hat{\otimes}} G)(X) = F(X) \otimes G(X)$. The tensor unit is the constant functor sending everything to $I \in \mathcal{C}$. We denote this structure $(\mathbf{SMC}(\mathcal{C}, \mathcal{C}), \hat{\otimes}, \hat{I})$.

(ii) By composition: The tensor product is composition of functors, and the tensor unit is the identity functor. We denote this structure $(\mathbf{SMC}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id})$.

Definition 4.3 Let $\mathcal{C}$ be a symmetric monoidal category, and let $\mathsf{m} = (R_{\mathsf{m}}, \mathsf{Cont}(\mathsf{m}), \mathsf{Weak}(\mathsf{m}))$ be a mode. An *exponential action* (of m on $\mathcal{C}$) is a functor $D \colon R_{\mathsf{m}}^{\mathrm{op}} \to \mathbf{SMC}(\mathcal{C}, \mathcal{C})$ equipped with the following additional structure.

(i) The structure of a colax monoidal functor with signature $(R_{\mathsf{m}}^{\mathrm{op}}, \cdot, 1) \to (\mathbf{SMC}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id})$ on D, i.e. natural transformations $\delta_{r,q} \colon D(r \cdot q) \to D(r) \circ D(q)$ and $\epsilon \colon D(1) \to \mathrm{id}$.

(ii) For $r, q \in \mathsf{Cont}(\mathsf{m})$, a natural transformation $c_{r,q} \colon D(r + q) \to D(r) \mathbin{\hat{\otimes}} D(q)$. Making the following diagram commute.

$$\begin{array}{ccccc}
D((r_1 + r_2) + r_3) & \xrightarrow{c_{r_1+r_2,r_3}} & D(r_1 + r_2) \mathbin{\hat{\otimes}} D(r_3) & \xrightarrow{c_{r_1,r_2} \hat{\otimes} \mathrm{id}} & (D(r_1) \mathbin{\hat{\otimes}} D(r_2)) \mathbin{\hat{\otimes}} D(r_3) \\
\| & & & & \downarrow \sim \\
D(r_1 + (r_2 + r_3)) & \xrightarrow{c_{r_1,r_2+r_3}} & D(r_1) \mathbin{\hat{\otimes}} D(r_2 + r_3) & \xrightarrow{\mathrm{id} \hat{\otimes} c_{r_2,r_3}} & D(r_1) \mathbin{\hat{\otimes}} (D(r_2) \mathbin{\hat{\otimes}} D(r_3))
\end{array}$$

Furthermore, if $\mathsf{Weak}(\mathsf{m})$ is true, a morphism $w \colon D(0) \to \hat{I}$. In this case, the transformations $c_{r,q}$ and w are required to assemble into a colax monoidal structure on the functor

$$r \mapsto D(r) \colon (\mathsf{Cont}(\mathsf{m})^{\mathrm{op}}, +, 0) \to \mathbf{SMC}(\mathcal{C}, \mathcal{C}).$$

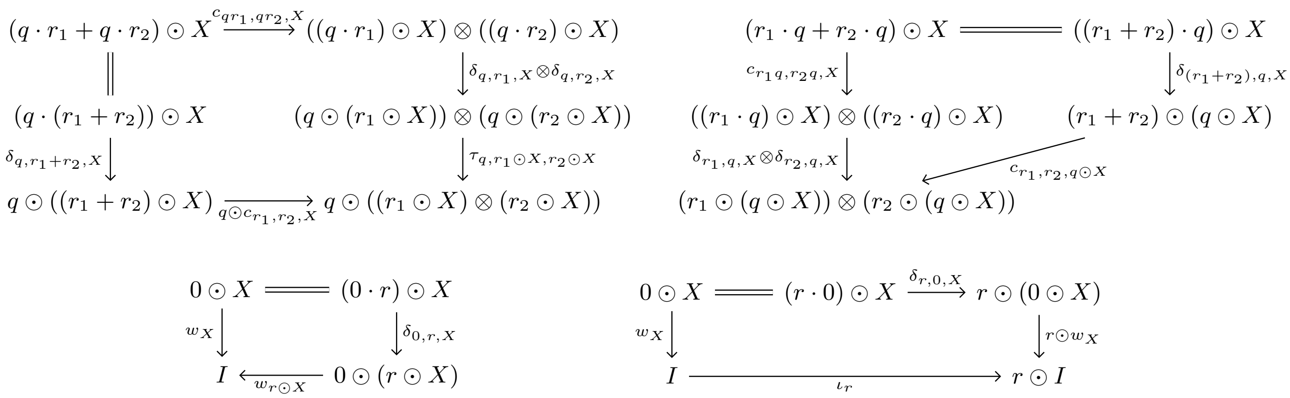

Fig. 4. Graded comonad coherence conditions

Note that the requirement that this functor is colax monoidal already includes the above diagram. We include that diagram explicitly, since we cannot require that this functor be colax monoidal in the absence of the morphism w.

These structures are related by the diagrams of Figure 4, which are required to commute whenever all morphisms in them are defined. For the first two diagrams this means that $r_1, r_2 \in \mathsf{Cont}(\mathsf{m})$ and for the last two diagrams this means that $\mathsf{Weak}(\mathsf{m})$ is true. We write $\tau_{r,X,Y} : D(r)(X) \otimes D(r)(Y) \to D(r)(X \otimes Y)$ and $\iota_r : I \to D(r)(I)$ for the strong monoidal structure on $D(r)$. Furthermore, the diagrams in Figure 4 use the infix notation $D(r)(X) =: r \odot X$, which we will use for the remainder of this text.

Example 4.4 We consider exponential actions of the modes given in Example 2.8. Let $\mathcal{C}$ be an SMCC.

(i) Any SMCC admits an exponential action of the initial mode L by setting $n \odot X = X^{\otimes n}$. Similarly, any SMCC admits an exponential action of the mode $(\{0,1\}, \{0\}, \mathsf{false})$, by setting $0 \odot X = I$ and $1 \odot X = X$.

(ii) An exponential action of the terminal mode U is equivalent to specifying the structure of a Linear Category [7] on $\mathcal{C}$. In other words, an SMCC with an exponential action of the terminal mode is precisely a model of Intuitionistic Linear Logic, with the exponential action modelling the of-course modality. If we choose $1 \odot (-) = \mathrm{id}$, the action equips $\mathcal{C}$ with "projections and diagonals" and hence $\mathcal{C}$ is a cartesian closed category. [4]

(iii) An exponential action of the mode R on $\mathcal{C}$ is a functor D together with diagonal maps $DX \to DX \otimes DX$. If we fix $D = \mathrm{id}$, then $\mathcal{C}$ becomes equipped with the structure of a *relevant category* [27]. Conversely, any relevant category admits an exponential action of R by setting $1 \odot (-) = \mathrm{id}$.

(iv) If the tensor unit I is the terminal object of $\mathcal{C}$, then $\mathcal{C}$ admits an exponential action of the mode A by setting $1 \odot (-) = \mathrm{id}$ and $0 \odot (-) = \hat{I}$. Conversely, an action of A on $\mathcal{C}$ with $1 \odot (-) = \mathrm{id}$ and $0 \odot (-) = \hat{I}$ exhibits the tensor unit as the terminal object of $\mathcal{C}$.

All in all, we see that known models of intuitionistic, linear, relevant and affine logic admit canonical exponential actions of the modes $\mathsf{U}, \mathsf{L}, \mathsf{R}$ and A respectively.

Definition 4.5 By a *symmetric lax monoidal adjunction* we mean a symmetric lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between SMCs together with a lax monoidal right adjoint G, such that the unit and counit of the adjunction are monoidal natural transformations. We concisely denote this by $F \dashv G : \mathcal{C} \to \mathcal{D}$. Recall that F is strong monoidal in this case [18].

Definition 4.6 Let $\varphi : \mathsf{m} \to \mathsf{n}$ be a morphism of modes, and let $\odot_{\mathsf{m}}$ and $\odot_{\mathsf{n}}$ be exponential actions of m, n on the SMCCs $\mathcal{C}_{\mathsf{m}}$ and $\mathcal{C}_{\mathsf{n}}$ respectively. A morphism of actions (over φ) consists of a symmetric lax

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[4] This result is folklore; A proof is given in Melliès [22].

$$\begin{array}{ccc} \varphi(1)\odot G(A) & \xrightarrow{\ell} & G(1\odot A) \\ \| & & \downarrow{\scriptstyle G(\epsilon)} \\ 1\odot G(A) & \xrightarrow[\epsilon]{} & G(A) \end{array} \qquad \begin{array}{ccc} 0\odot G(A) & \xrightarrow{\ell} & G(0\odot A) \\ {\scriptstyle w_{G(A)}}\downarrow & & \downarrow{\scriptstyle G(w_A)} \\ I & \longrightarrow & G(I) \end{array}$$

$$\begin{array}{ccc} (\varphi(q)\cdot\varphi(r))\odot G(A) & = & \varphi(qr)\odot G(A) \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \ell} \\ \varphi(q)\odot\varphi(r)\odot G(A) & & G(qr\odot A) \\ {\scriptstyle \varphi(q)\odot\ell}\downarrow & & \downarrow{\scriptstyle G(\delta)} \\ \varphi(q)\odot G(r\odot A) & \xrightarrow{\ell} & G(q\odot r\odot A) \end{array} \qquad \begin{array}{ccc} (\varphi(q)+\varphi(r))\odot G(A) & = & \varphi(q+r)\odot G(A) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \ell} \\ \varphi(q)\odot G(A)\otimes\varphi(r)\odot G(A) & & G((q+r)\odot A) \\ {\scriptstyle \ell\otimes\ell}\downarrow & & \downarrow{\scriptstyle G(c)} \\ G(q\odot A)\otimes G(r\odot A) & \longrightarrow & G(q\odot A\otimes r\odot A) \end{array}$$

Fig. 5. Coherence laws on morphisms of exponential actions. Unlabelled arrows are the lax monoidal structure on G.

monoidal adjunction $F \dashv G\colon \mathcal{C}_{\mathsf{n}} \to \mathcal{C}_{\mathsf{m}}$ together with a monoidal natural transformation

$$\ell_{r,A}\colon \varphi(r) \odot_{\mathsf{n}} G(A) \to G(r \odot_{\mathsf{m}} A)$$

satisfying the coherence conditions given in Figure 5. We define another natural transformation $\mu_{r,A}\colon F(\varphi(r)\odot A) \to r \odot F(A)$ as the transpose of the composite

$$\varphi(r)\odot A \xrightarrow{\varphi(r)\odot\eta} \varphi(r)\odot GFA \xrightarrow{\ell} G(r\odot FA).$$

Morphisms between exponential actions compose in a straightforward way, giving rise to a category **ExpAct** of exponential actions, whose objects are triples $(\mathsf{m},\mathcal{C},\odot)$ consisting of a mode m, an SMCC $\mathcal{C}$ and an exponential action of m on $\mathcal{C}$, and whose morphisms are quadruples (φ,F,G,ℓ), where φ is a morphism of modes and (F,G,ℓ) is a morphism of actions over φ. We endow **ExpAct** with the structure of a 2-category: A 2-cell between (φ,F,G,ℓ) and (φ,F',G',ℓ') (note that the underlying morphisms of modes are required to be identical) is a natural isomorphism $\alpha\colon G\to G'$ which respects the lineators in the obvious way:

$$\begin{array}{ccc} \varphi(r)\odot GA & \xrightarrow{\ell} & G(r\odot A) \\ {\scriptstyle \varphi(r)\odot\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \varphi(r)\odot G'A & \xrightarrow[\ell']{} & G'(r\odot A) \end{array}$$

Remark 4.7 The terminology *lineator* is borrowed from the theory of actegories [sic]. In fact, our notion of morphisms between actions is adapted from the notion of morphisms between actegories. See e.g. Cappucci and Gavranović [9].

Remark 4.8 Let $(\varphi,F,G,\ell)\colon (\mathsf{m},\mathcal{C}_{\mathsf{m}},\odot_{\mathsf{m}}) \to (\mathsf{n},\mathcal{C}_{\mathsf{n}},\odot_{\mathsf{n}})$ be a morphism in **ExpAct**. Transposing ℓ along the adjunction $F\dashv G$ yields a natural transformation $F(\varphi(-)\odot_{\mathsf{n}} G(=)) \to (-)\odot_{\mathsf{m}}(=)$. Katsumata proved that the domain of this transformation satisfies the coherence conditions for exponential actions [17]. In Katsumata's setting, this functor is in fact a graded linear exponential comonad. Here, it may fail to be an exponential action since G is not required to be strong monoidal. Thus, ℓ is equivalent to a comparison morphism of the two actions on $\mathcal{C}_{\mathsf{m}}$ that naturally arise from the adjunction $F\dashv G$.

Lemma 4.9 *The coherence conditions on ℓ given in Figure 5 are equivalent to the analogous coherence conditions on μ given in Figure 6.*

Proof. Straightforward manipulation of diagrams, using the fact that ℓ can be written as the composite $\varphi(r)\odot G(A) \xrightarrow{\eta} GF(\varphi(r)\odot G(A)) \xrightarrow{G(\mu)} G(r\odot FG(A)) \xrightarrow{G(r\odot\epsilon)} G(r\odot A)$. Here ϵ is the counit of the adjunction $F\dashv G$. $\square$

$$\begin{array}{ccc} F(\varphi(1)\odot A) & \xrightarrow{\mu} & 1\odot FA \\ \| & & \downarrow{\scriptstyle \epsilon} \\ F(1\odot A) & \xrightarrow[F\epsilon]{} & F(A) \end{array} \qquad \begin{array}{ccc} F(\varphi(0)\odot A) & \xrightarrow{\mu} & 0\odot FA \\ {\scriptstyle Fw}\downarrow & & \downarrow{\scriptstyle w} \\ F(I) & \xrightarrow[u^{-1}]{} & I \end{array}$$

$$\begin{array}{ccc} F(\varphi(q)\varphi(r)\odot A) & = & F(\varphi(qr)\odot A) \\ {\scriptstyle F(\delta)}\downarrow & & \downarrow \\ F(\varphi(q)\odot\varphi(r)\odot A) & & qr\odot F(A) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \delta} \\ q\odot F(\varphi(r)\odot A) & \xrightarrow[q\odot\mu]{} & q\odot r\odot FA \end{array} \qquad \begin{array}{ccc} F((\varphi(q)+\varphi(r))\odot A) & = & F(\varphi(q+r)\odot A) \\ {\scriptstyle F(c)}\downarrow & & \downarrow{\scriptstyle \mu} \\ F(\varphi(q)\odot A\otimes\varphi(r)\odot A) & & (q+r)\odot FA \\ {\scriptstyle m^{-1}}\downarrow & & \downarrow{\scriptstyle c} \\ F(\varphi(q)\odot A)\otimes F(\varphi(r)\odot A) & \xrightarrow[\mu\otimes\mu]{} & q\odot FA\otimes r\odot FA \end{array}$$

Fig. 6. Coherence laws on μ.

Lemma 4.10 *Let (G, F, ℓ) and (G', F', ℓ') be morphisms of actions over φ, and $\alpha\colon G \to G'$ a 2-cell in* **ExpAct**. *Then α induces a natural isomorphism $\beta\colon F' \to F$ which respects μ in the obvious way:*

$$\begin{array}{ccc} F'(\varphi(q)\odot A) & \xrightarrow{\mu} & q\odot F'(A) \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle q\odot\beta} \\ F(\varphi(q)\odot A) & \xrightarrow[\mu']{} & q\odot F(A) \end{array}$$

Definition 4.11 Recall that Grass is parametrized by a functor $d\colon S \to \mathbf{Mode}$ where S is some preordered set. We have a projection functor $p\colon \mathbf{ExpAct} \to \mathbf{Mode}$ given by $(\mathsf{m}, \mathcal{C}, \odot) \mapsto \mathsf{m}$. A categorial model for Grass consists of a pseudofunctor $D\colon S \to \mathbf{ExpAct}$ such that $p \circ D = d$. Spelling this out, this amounts to specifying the following data: For each mode m, an SMCC $\mathcal{C}_{\mathsf{m}}$ together with an exponential action $\odot_{\mathsf{m}}$ of m on $\mathcal{C}_{\mathsf{m}}$. And whenever $\mathsf{m} \leq \mathsf{n}$, a morphism of exponential actions $(\varphi_{\mathsf{m,n}}, F^{\mathsf{n}}_{\mathsf{m}}, G^{\mathsf{n}}_{\mathsf{m}}, \ell^{\mathsf{n}}_{\mathsf{m}})\colon (\mathsf{m}, \mathcal{C}_{\mathsf{m}}, \odot_{\mathsf{m}}) \to (\mathsf{n}, \mathcal{C}_{\mathsf{n}}, \odot_{\mathsf{n}})$ where $\varphi_{\mathsf{m,n}}$ is the morphism of grade algebras specified in Definition 3.1 along with a coherent system of natural isomorphisms $G^{\mathsf{m}_3}_{\mathsf{m}_2} \circ G^{\mathsf{m}_2}_{\mathsf{m}_1} \cong G^{\mathsf{m}_3}_{\mathsf{m}_1}$ whenever $\mathsf{m}_1 \leq \mathsf{m}_2 \leq \mathsf{m}_3$ which are compatible with the lineators.

We write $u\colon I \to F^{\mathsf{m}}_{\mathsf{n}}(I)$ and $m\colon F^{\mathsf{m}}_{\mathsf{n}}(A) \otimes F^{\mathsf{m}}_{\mathsf{n}}(B) \to F^{\mathsf{m}}_{\mathsf{n}}(A \otimes B)$ for the strong monoidal structure on the functors $F^{\mathsf{m}}_{\mathsf{n}}$. We denote internal hom-objects by $X \multimap Y$. We write $ev\colon (X \multimap Y) \otimes X \to Y$ for the counit of the of the adjunction $(-) \otimes X \dashv X \multimap (-)$. Given $e\colon X \otimes Y \to Z$ we write $\lambda e\colon X \to (Y \multimap Z)$ for the transpose of e along this adjunction.

Example 4.12 Let $F \dashv G\colon \mathcal{C} \to \mathcal{L}$ be a symmetric lax monoidal adjunction. Suppose that L acts on $\mathcal{L}$ as described in Example 4.4, and that $\mathcal{C}$ admits an exponential action by a mode $\mathsf{m} = (R, R, \mathsf{true})$ for some grade algebra R. We promote this adjunction to a morphism of exponential actions with lineator

$$\varphi(n) \odot G(A) = (\varphi(1) + \ldots + \varphi(1)) \odot G(A) \xrightarrow{l} G(A)^{\otimes n} \xrightarrow{z} G(A^{\otimes n})$$

where the first arrow, l, is c (resp. w if $n = 0$) and the second arrow, z, is the monoidal structure on G. Therefore, any model of mGL [32] is a model of Grass instantiated with the modes $\mathsf{L} < \mathsf{m}$. When $\mathsf{m} = \mathsf{U}$, $\mathcal{C}$ is cartesian closed, and U acts on $\mathcal{C}$ by $1 \odot (-) = \mathrm{id}$, as described in Example 4.4, we recover the categorical situation of LNL [4].

Definition 4.13 Recall the rules for well-formed types from Figure 1. We assign each type $A \in \mathsf{Type}(\mathsf{m})$

Name	Type	Isomorphism?
δ	$(rq) \odot X \to r \odot (q \odot X)$	no
ϵ	$1 \odot X \to X$	no
τ	$(r \odot X) \otimes (r \odot Y) \to r \odot (X \otimes Y)$	yes
ι	$I \to r \odot I$	yes
c	$(r+q) \odot X \to (r \odot X) \otimes (q \odot X)$	no
w	$0 \odot X \to I$	no
m	$F(X) \otimes F(Y) \to F(X \otimes Y)$	yes
u	$I \to F(I)$	yes
μ	$F(\varphi_{\mathsf{n},\mathsf{m}}(q) \odot X) \to q \odot F^{\mathsf{m}}_{\mathsf{n}}(X)$	no

Fig. 7. Summary of named natural transformations in the categorical model for GRASS

an object of $[\![A]\!] \in \mathcal{C}_{\mathsf{m}}$ by induction on the proof of $A \in \mathsf{Type}(\mathsf{m})$.

$$\text{UNIT}\ \frac{}{[\![\mathbf{I}_{\mathsf{m}}]\!] = I \in \mathcal{C}_{\mathsf{m}}} \qquad \text{PAIR}\ \frac{A \in \mathsf{Type}(\mathsf{m}) \qquad B \in \mathsf{Type}(\mathsf{m})}{[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!] \in \mathcal{C}_{\mathsf{m}}} \qquad \text{FUN}\ \frac{\mathsf{n} \le \mathsf{m} \qquad q \in R_{\mathsf{m}} \qquad A \in \mathsf{Type}(\mathsf{m}) \qquad B \in \mathsf{Type}(\mathsf{n})}{[\![A^{q:\mathsf{m}} \multimap B]\!] = F^{\mathsf{m}}_{\mathsf{n}}(q \odot [\![A]\!]) \multimap [\![B]\!] \in \mathcal{C}_{\mathsf{n}}}$$

$$\text{DROP}\ \frac{\mathsf{n} \le \mathsf{m} \qquad q \in R_{\mathsf{m}} \qquad A \in \mathsf{Type}(\mathsf{m})}{[\![\downarrow^{q}_{\mathsf{n}\le\mathsf{m}} A]\!] = F^{\mathsf{m}}_{\mathsf{n}}(q \odot [\![A]\!]) \in \mathcal{C}_{\mathsf{n}}} \qquad \text{DROP}\ \frac{\mathsf{m} \le \mathsf{n} \qquad A \in \mathsf{Type}(\mathsf{m})}{[\![\uparrow_{\mathsf{m}\le\mathsf{n}} A]\!] = G^{\mathsf{n}}_{\mathsf{m}}[\![A]\!] \in \mathcal{C}_{\mathsf{n}}}$$

We will often omit the brackets $[\![-]\!]$ writing A instead of $[\![A]\!]$.

Definition 4.14 Let $\rho = r_1, \ldots, r_k$ and $\mathsf{M} = \mathsf{m}_1, \ldots, \mathsf{m}_k$ and $\Gamma = x_1 : A_1, \ldots, x_k : A_k$, and $\mathsf{M} \ge \mathsf{m}$. Define

$$[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} := \bigotimes_{i=1}^{k} [\![\downarrow^{r_i}_{\mathsf{m}\le\mathsf{m}_i} A_i]\!] = \bigotimes_{i=1}^{k} F^{\mathsf{m}_i}_{\mathsf{m}}(r_i \odot A_i)$$

Note that there is a canonical isomorphism $F^{\mathsf{m}}_{\mathsf{n}}([\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}}) \cong [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}}$ whenever $\mathsf{n} \le \mathsf{m}$.

Definition 4.15 In the situation of Definition 4.14, let $q \in R_{\mathsf{m}}$ and let $t\colon [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \to A$ be a morphism. Define $q \cdot t\colon [\![q\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \to q \odot A$ as

$$\bigotimes_{i=1}^{k} F^{\mathsf{m}_i}_{\mathsf{m}}(q \cdot r_i) \odot A_i \xrightarrow{\tau \circ \bigotimes_i (\mu \circ F^{\mathsf{m}_i}_{\mathsf{m}} \delta)} q \odot \bigotimes_{i=1}^{k} F^{\mathsf{m}_i}_{\mathsf{m}}(r_i \odot A_i) \xrightarrow{q \odot t} q \odot A$$

for $k > 0$. When $k = 0$, define $q \cdot t = (q \odot t) \circ \iota\colon I_{\mathsf{m}} \to q \odot I_{\mathsf{m}} \to q \odot A$.

Definition 4.16 We assign each judgment $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t\colon A$ a morphism $[\![t]\!]\colon [\![\rho \mid \mathsf{M} \odot \Gamma]\!] \to [\![A]\!]$ in $\mathcal{C}_{\mathsf{m}}$. This assignment is specified in Figure 8 by induction on the derivation of t.

Theorem 4.17 (Soundness) *β- and η-conversions are sound with respect to the categorical semantics:*

(i) *If $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t_1 \equiv_\beta t_2\colon A$ then $[\![t_1]\!] = [\![t_2]\!]$ in $\mathcal{C}_{\mathsf{m}}$.*

(ii) *If $\rho \mid \mathsf{M} \odot \Gamma \vdash_{\mathsf{m}} t_1 \equiv_\eta t_2\colon A$ then $[\![t_1]\!] = [\![t_2]\!]$ in $\mathcal{C}_{\mathsf{m}}$.*

Proof. Given in §§C.2, C.3 of the appendix [16]. □

5 Conclusion, related and future work

We defined a graded logic which allows grades from multiple modes to exist in a single system. Furthermore, we have generalized the graded weakening and contraction rules to obtain more fine-grained control in the graded setting. This allowed us to recover non-graded substructural logics in an entirely graded setting. We

$$\textsc{Var}\quad \frac{}{F^{\mathsf{m}}_{\mathsf{m}}(1 \odot A) \cong 1 \odot A \xrightarrow{e} A}$$

$$\textsc{Weak}\quad \frac{\mathsf{Weak}(\mathsf{l}) \qquad \mathsf{m} \leq \mathsf{l} \qquad B \in \mathsf{Type}(\mathsf{l}) \qquad [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \xrightarrow{t} A}{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \otimes F^{\mathsf{l}}_{\mathsf{m}}(0 \odot B) \xrightarrow{t \otimes (u^{-1} \circ F(w))} A \otimes I \cong A}$$

$$\textsc{Sub}\quad \frac{(r_i \leq q_i)_i \qquad \bigotimes_i F^{\mathsf{m}_i}_{\mathsf{m}}(r_i \odot X_i) \xrightarrow{t} A}{\bigotimes_i F^{\mathsf{m}_i}_{\mathsf{m}}(q_i \odot X_i) \xrightarrow{F^{\mathsf{m}_i}_{\mathsf{m}}((r_i \leq q_i) \odot X_i)} \bigotimes_i F^{\mathsf{m}_i}_{\mathsf{m}}(r_i \odot X_i) \xrightarrow{t} A}$$

$$\textsc{Cont}\quad \frac{r_1, r_2 \in \mathsf{Cont}(\mathsf{m}) \qquad [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \otimes F^{\mathsf{m}}_{\mathsf{n}}(r_1 \odot A) \otimes F^{\mathsf{m}}_{\mathsf{n}}(r_2 \odot A) \xrightarrow{t} B}{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \otimes F^{\mathsf{m}}_{\mathsf{n}}((r_1 + r_2) \odot A) \xrightarrow{\mathrm{id} \otimes (m^{-1} \circ F^{\mathsf{m}}_{\mathsf{n}}(c))} [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \otimes F^{\mathsf{m}}_{\mathsf{n}}(r_1 \odot A) \otimes F^{\mathsf{m}}_{\mathsf{n}}(r_2 \odot A) \xrightarrow{t} B}$$

$$\textsc{Unit.i}\quad \frac{}{I_{\mathsf{m}} \xrightarrow{\mathrm{id}} I_{\mathsf{m}}}$$

$$\textsc{Unit.e}\quad \frac{[\![\sigma \mid \mathsf{N} \odot \Delta]\!]_{\mathsf{m}} \xrightarrow{e} I_{\mathsf{m}} \qquad [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \xrightarrow{t} A}{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \otimes [\![q\sigma \mid \mathsf{N} \odot \Delta]\!]_{\mathsf{m}} \xrightarrow{t \otimes (\iota^{-1} \circ (q \cdot e))} A \otimes I_{\mathsf{m}} \cong A}$$

$$\textsc{Fun.i}\quad \frac{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \otimes F^{\mathsf{m}}_{\mathsf{n}}(r \odot A) \xrightarrow{e} B}{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \xrightarrow{\lambda e} F^{\mathsf{m}}_{\mathsf{n}}(r \odot A) \multimap B}$$

$$\textsc{Fun.e}\quad \frac{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \xrightarrow{t} F^{\mathsf{m}}_{\mathsf{n}}(q \odot A) \multimap B \qquad [\![\sigma \mid \mathsf{N} \odot \Delta]\!]_{\mathsf{m}} \xrightarrow{e} A}{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \otimes F^{\mathsf{m}}_{\mathsf{n}}[\![\sigma \mid \mathsf{N} \odot \Delta]\!]_{\mathsf{m}} \xrightarrow{t \otimes F^{\mathsf{m}}_{\mathsf{n}}(q \cdot e)} (F^{\mathsf{m}}_{\mathsf{n}}(q \odot A) \multimap B) \otimes F^{\mathsf{m}}_{\mathsf{n}}(q \odot A) \xrightarrow{ev} B}$$

$$\textsc{Pair.i}\quad \frac{[\![\rho_1 \mid \mathsf{M}_1 \odot \Gamma_1]\!]_{\mathsf{m}} \xrightarrow{a_1} A_1 \qquad [\![\rho_2 \mid \mathsf{M}_2 \odot \Gamma_2]\!]_{\mathsf{m}} \xrightarrow{a_2} A_2}{[\![\rho_1 \mid \mathsf{M}_1 \odot \Gamma_1]\!]_{\mathsf{m}} \otimes [\![\rho_2 \mid \mathsf{M}_2 \odot \Gamma_2]\!]_{\mathsf{m}} \xrightarrow{a_1 \otimes a_2} A_1 \otimes A_2}$$

$$\textsc{Pair.e}\quad \frac{[\![\sigma \mid \mathsf{N} \odot \Delta]\!]_{\mathsf{m}} \xrightarrow{e} A_1 \otimes A_2 \qquad [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \otimes F^{\mathsf{m}}_{\mathsf{n}}(q \odot A_1) \otimes F^{\mathsf{m}}_{\mathsf{n}}(q \odot A_2) \xrightarrow{t} B}{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \otimes F^{\mathsf{m}}_{\mathsf{n}}[\![q\sigma \mid \mathsf{N} \odot \Delta]\!]_{\mathsf{m}} \xrightarrow{\mathrm{id} \otimes (m^{-1} \circ F^{\mathsf{m}}_{\mathsf{n}}(\tau^{-1} \circ (q \cdot e)))} [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \otimes F^{\mathsf{m}}_{\mathsf{n}}(q \odot A_1) \otimes F^{\mathsf{m}}_{\mathsf{n}}(q \odot A_2) \xrightarrow{t} B}$$

$$\textsc{Raise.i}\quad \frac{\mathsf{m} \leq \mathsf{n} \leq \mathsf{M} \qquad [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \xrightarrow{e} A}{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \xrightarrow{\mathrm{unit}} G^{\mathsf{n}}_{\mathsf{m}} F^{\mathsf{m}}_{\mathsf{n}} [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \cong G^{\mathsf{n}}_{\mathsf{m}} [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \xrightarrow{G^{\mathsf{n}}_{\mathsf{m}} e} G^{\mathsf{n}}_{\mathsf{m}} A}$$

$$\textsc{Raise.e}\quad \frac{\mathsf{m} \leq \mathsf{n} \qquad [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \xrightarrow{e} G^{\mathsf{n}}_{\mathsf{m}} A}{F^{\mathsf{m}}_{\mathsf{n}} [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{n}} \xrightarrow{F^{\mathsf{m}}_{\mathsf{n}} e} F^{\mathsf{m}}_{\mathsf{n}} G^{\mathsf{n}}_{\mathsf{m}} A \xrightarrow{\mathrm{counit}} A}$$

$$\textsc{Drop.i}\quad \frac{\mathsf{n} \leq \mathsf{m} \qquad q \in R_{\mathsf{m}} \qquad [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \xrightarrow{e} A}{F^{\mathsf{m}}_{\mathsf{n}} [\![q\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{m}} \xrightarrow{F^{\mathsf{m}}_{\mathsf{n}}(q \cdot e)} F^{\mathsf{m}}_{\mathsf{n}}(q \odot A)}$$

$$\textsc{Drop.e}\quad \frac{\mathsf{l} \leq \mathsf{n} \leq \mathsf{m} \qquad [\![\sigma \mid \mathsf{N} \odot \Delta]\!]_{\mathsf{n}} \xrightarrow{e} F^{\mathsf{m}}_{\mathsf{n}}(q \odot A) \qquad [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{l}} \otimes F^{\mathsf{m}}_{\mathsf{l}}(q \odot A) \xrightarrow{t} B}{[\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{l}} \otimes F^{\mathsf{n}}_{\mathsf{l}} [\![\sigma \mid \mathsf{N} \odot \Delta]\!]_{\mathsf{n}} \xrightarrow{\mathrm{id} \otimes F^{\mathsf{n}}_{\mathsf{l}} e} [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{l}} \otimes F^{\mathsf{n}}_{\mathsf{l}} F^{\mathsf{m}}_{\mathsf{n}}(q \odot A) \cong [\![\rho \mid \mathsf{M} \odot \Gamma]\!]_{\mathsf{l}} \otimes F^{\mathsf{m}}_{\mathsf{l}}(q \odot A) \xrightarrow{t} B}$$

Fig. 8. Grass interpretation into a categorical model

gave an account of the categorical semantics of our logic based on graded comonads. Our logic consists of multiple graded systems connected by morphisms of modes, and this design is mirrored in the categorical semantics. We introduced a novel notion of morphisms of graded comonads, and showed how it connects to the existing literature. We saw that categorical models of substructural logics provide categorical models for the modes which recover such logics at the syntactic level.

Combined systems

Systems which combine logics permitting different sets of structural rules originate in Benton's Linear-non-Linear Logic (LNL) [4], a system which combines linear and intuitionistic logics. A similar result was achieved for graded and linear logics in mGL of Vollmer et al. [32]. In the non-graded world, LNL was further generalized by Pruiksma et al. into Adjoint Logic [28], which allows an arbitrary collection of logics with different structural rules (relevant, affine, linear, intuitionistic) to exist in one system. The present work initially started out as a "graded Adjoint Logic," and thus bears many similarities to Adjoint Logic. For example, our mode system and modal operators $\uparrow_{\mathsf{n} \le \mathsf{m}} A$ and $\downarrow^{q}_{\mathsf{n} \le \mathsf{m}} A$ are direct adaptations of the mode system and modal operators presented in Adjoint Logic. All of the systems mentioned here have one observation in common: Terms from logics with fewer structural rules may depend on variables permitting more structural rules, but not vice-versa. This invariant is also present in our work, and we leverage it to ensure that reuse of more structured variables is well-behaved in less structured modes.

Heterogenously graded systems

Systems where grades can be drawn from more than one grade algebra have been studied for subsets of Java by Bianchini et al. [5,6] as well as Giannini and Duso [14]. We refer to these works collectively as *Graded Java*. Modes in GRASS correspond to *kinds* of coeffects in Graded Java. Both modes and kinds are arranged in a preorder, and both systems use morphisms of grade algebras to combine (i.e. add or multiply) grades from different algebras.

We highlight some differences: Graded Java reduces the problem of tracking multiple kinds of grades in one program to the case of a single grade algebra by equipping the disjoint union $\bigsqcup_k R_k$ with an appropriate grade algebra structure, where k ranges over the set of kinds of coeffects. The set of kinds is required to have joins, and if $r_1 \in R_{k_1}$ and $r_2 \in R_{k_2}$, then $r_1 + r_2 \in R_{k_1 \vee k_2}$ and similarly for multiplication. This can lead to a loss of information, e.g. when $k_1 \vee k_2$ is the terminal kind $k_\top$ with $R_{k_\top} = \top$ the teminal semiring. This means that a variable can be marked unrestricted in a term, even though it should have non-trivial usage restrictions in each of its subterms. In contrast to this, GRASS's typing rules ensure that additions only happen between grades from the same mode, and multiplications only happen when the modes are comparable in the preorder, thus avoiding the need for joins. Furthermore each variable in GRASS belongs to a unique mode, and its grades are always guaranteed to be drawn from that mode. Hence there is no risk of the loss of information mentioned above. We leave it to future work to investigate if/how our approach can be implemented in a system like Graded Java.

Categorical semantics of graded modal types

Many existing graded type systems have been given categorical semantics [8,12,13,25,26]. These semantics were all somewhat similar, employing a notion of *graded linear exponential comonads* (though terminology varied). Later, Katsumata [17] gave a comprehensive account of graded linear exponential comonads. Katsumata's work also serves as the basis for the exponential actions defined in the present text. Beyond the changes that were necessary to account for our modified weakening and contraction rules, there is one other difference between our presentation and that of Katsumata: We require that $r \odot (-)$ be a strong monoidal functor, while the categorical semantics in the work cited here all require this functor only to be lax monoidal. We attribute this difference to the fact that strong monoidality of $r \odot (-)$ is required for the interpretation of the tensor product type, which was not included in the above works. Cateogrical semantics were given to mGL by Vollmer et al. [32]. This type system does have a tensor product, and they require the functors $r \odot (-)$ to be strict monoidal, i.e. the morphism $(r \odot A) \otimes (r \odot B) \to r \odot (A \otimes B)$ must be an identity. We have shown here that the strictness is not necessary for the interpretation of the graded tensor product. Finally, Atkey [2] has given categorical semantics to a dependently typed graded system using *Quantitative Categories with Families*, an approach which is not based on graded comonads.

Future work

One future direction is to use the lessons learned here about the categorical semantics of graded types towards a categorical semantics of graded *dependent* types which is based on graded comonads. One problem in this area is that uses of terms inside of types are considered computationally irrelevant, and therefore exempt from grading. In other words, types are treated as intuitionistic, but terms must still be graded. The categorical semantics presented here are able to capture graded and intuitionistic variables existing in the same system, by instantiating GRASS with any one mode for the graded system, together with the terminal mode to recover intuitionistic logic. We hope to use this as a starting point for the categorical semantics of graded dependent types.

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