An explicit shuffle construction of the homotopy span model of linear logic

Paul-André Melliès[a] Federico Olimpieri[b] Lionel Vaux Auclair[b]

[a] *CNRS, Université Paris Cité, Inria*

[b] *Aix-Marseille Université*

Abstract

We give a direct and explicit construction of the homotopy span model of linear logic originally formulated by Melliès using abstract homotopy theory in the Quillen category **Cat** of small categories equipped with its natural model structure. Following the principles of homotopy theory, the formulas and proofs of linear logic are interpreted in the model as small categories up to categorical equivalence, the notion of weak homotopy equivalence underlying the natural Quillen model structure on **Cat**. Here, we explain how the original interpretation of proofs as *fibrant spans* can be replaced by a more liberal interpretation of proofs as *separately fibrant spans*. This relaxation from fibrant spans to separately fibrant spans enables us to give a simple and intuitive description of the homotopy span model, where the axiom and cut links are interpreted as identity spans, and where the symmetries of the exponential modality are captured by a shuffle structure on the interpretation of proofs.

1 Introduction

The relational semantics of linear logic [18,8,9] has become today a fundamental tool to investigate and to analyze the structure of higher-order computations. It provides at the same time an abstract mathematical framework which played a central role in the discovery of differential linear logic [9,11,10], and a concrete computational interpretation through its strong connection to intersection types [7,2,3,32] and higher-order model checking [26,19].

About twenty years ago, Fiore, Gambino, Hyland and Winskel introduced a categorified version of the relational semantics based on distributors, motivated by an unexpected link with Joyal's theory of combinatorial species [13]. This step of categorification is needed in order to obtain a concrete interpretation of linear logic, where the elements of the denotation of a proof coincide with the (equivalence classes of) derivation trees typing the proof in a specific intersection type system [29].

The model of generalized species has been a source of inspiration for a new generation of categorified models based on distributors [17,16,12]. An alternative way to categorify relations between sets is to refine them into functorial spans between categories. Melliès explained how to construct a functorial span model of linear logic [27] based on template games, where functorial spans are considered up to homotopy in the natural Quillen model structure. Following the homotopy span model, other approaches to functorial spans based on biorthogonality [5] or ∞-categories [20] have been developed. In this paper, we will give a more explicit presentation of the homotopy span model, exploiting the key notion of shuffle permutation on the letters of a finite word. We start by recalling the homotopy span model and its relationship with the generalized species interpretation of linear logic.

Two alternative categorifications of relations

Given two sets A and B, every binary relation $R \subseteq A \times B$ may be equivalently described as a function $A \times B \to \mathbf{2}$ where $\mathbf{2} = \{\mathbf{true}, \mathbf{false}\}$ denotes the set of booleans, or as a span $A \leftarrow S \to B$ consisting of a set S and of two jointly injective functions $\mathbf{source} : S \to A$ and $\mathbf{target} : S \to B$. Each point of view induces a specific way to *categorify* the notion of binary relation, and to refine the category $\mathbf{Rel}$ of sets and relations into a bicategory $\mathcal{W}$ of *categorified relations* between small categories A and B.

In the first approach, one defines the bicategory $\mathcal{W} = \mathbf{Dist}$ of small categories and *distributors*, where a distributor $M : A \longrightarrow\!\!\!\!\!\!\!\!+\;\; B$ is defined as a set-valued functor $M : A \times B^{op} \to \mathbf{Set}$, composed by a coend formula, and where a 2-cell $\theta : M \Rightarrow N$ between distributors $M, N : A \longrightarrow\!\!\!\!\!\!\!\!+\;\; B$ is defined as a natural transformation between the underlying functors.

In the second approach, one defines the bicategory $\mathcal{W} = \mathbf{Span}$ of small categories and *spans of functors* where a span $\mathcal{S} : A \longrightarrow\!\!\!\!\!\!\!\!+\;\; B$ is defined as a triple $\mathcal{S} = (S, \mathbf{source}, \mathbf{target})$ consisting of a small category S called the *support category* of $\mathcal{S}$, and of a pair of functors

$$A \xleftarrow{\ \mathbf{source}\ } S \xrightarrow{\ \mathbf{target}\ } B \tag{1}$$

composed by computing pullbacks in $\mathbf{Cat}$, and where a 2-cell $\theta : \mathcal{S} \Rightarrow \mathcal{T}$ (called a *simulation* 2-cell) between spans $\mathcal{S}, \mathcal{T} : A \longrightarrow\!\!\!\!\!\!\!\!+\;\; B$ is defined as a functor $\theta : S \to T$ between the underlying support categories, making the diagram commute:

$$\begin{array}{ccccc} & \mathbf{source}_S & S & \mathbf{target}_S & \\ A & \swarrow\!\!\!\nwarrow & \downarrow \theta & \searrow\!\!\!\nearrow & B\,. \\ & \mathbf{source}_T & T & \mathbf{target}_T & \end{array} \tag{2}$$

Note that $\mathbf{Span}$ is the usual span construction of a bicategory on the category $\mathbf{Cat}$ with pullbacks. Interestingly, both bicategories $\mathbf{Dist}$ and $\mathbf{Span}$ are $*$-autonomous (in fact compact closed) with tensor product $A, B \mapsto A \otimes B$ defined as categorical product $A, B \mapsto A \times B$, as well as cartesian and cocartesian with sum $A, B \mapsto A \oplus B$ and product $A, B \mapsto A \& B$ both defined as categorical sum $A, B \mapsto A + B$. For that reason, $\mathbf{Dist}$ and $\mathbf{Span}$ define models of the multiplicative additive fragment MALL of linear logic.

Similitudes and differences between Dist and Span.

Before reviewing how the bicategories $\mathbf{Dist}$ and $\mathbf{Span}$ were equipped in [13,27] with an interpretation of the exponential modality $A \mapsto\, !A$ to obtain a model of linear logic, we find useful to mention that both categorifications share an important formal property: namely, each functor $F : A \to B$ induces an adjunction

$$L_F \quad : \quad A \underset{\longleftarrow\!\!\!\!\!\!\!+\;\;}{\overset{\longrightarrow\!\!\!\!\!\!\!+\;\;}{\perp}} B \quad : \quad R_F$$

in the bicategories $\mathcal{W} = \mathbf{Dist}$ and $\mathcal{W} = \mathbf{Span}$. In $\mathbf{Dist}$, the adjunction is defined as follows:

$$\begin{array}{rcl} L_F & = & (a, b) \mapsto B(b, Fa) \ : \ A \times B^{op} \longrightarrow \mathbf{Set} \\ R_F & = & (b, a) \mapsto B(Fa, b) \ : \ B \times A^{op} \longrightarrow \mathbf{Set} \end{array}$$

while the adjunction $L_F \dashv R_F$ of spans in $\mathbf{Span}$ is defined as follows

$$L_F \;=\; A \xleftarrow{\ \mathbf{id}_A\ } A \xrightarrow{\ F\ } B \qquad\qquad R_F \;=\; B \xleftarrow{\ F\ } A \xrightarrow{\ \mathbf{id}_A\ } A\,.$$

Note that if we consider the category $\mathbf{Cat}$ as the 2-category $\mathbf{Cat}^{=}$ with identity 2-cells only, then the construction $F \mapsto L_F$ is functorial in both cases, in the sense that it defines a weak 2-functor between bicategories (also called a pseudofunctor) $L : \mathbf{Cat}^{=} \to \mathcal{W}$ for both **Dist** and **Span**. On the other hand, one key difference between **Dist** and **Span** is that if, instead of $\mathbf{Cat}^{=}$, we consider **Cat** as the 2-category $\mathbf{Cat}^{\Rightarrow}$ with all natural transformations as 2-cells, then $L : \mathbf{Cat}^{=} \to \mathcal{W}$ extends to a weak 2-functor $L : \mathbf{Cat}^{\Rightarrow} \to \mathcal{W}$ in the case $\mathcal{W} = \mathbf{Dist}$ but not in the case $\mathcal{W} = \mathbf{Span}$. Indeed, a 2-cell from L_F to L_G in **Span** is given by an endofunctor $\theta : A \to A$ making the diagram (2) commute with both $\mathbf{source}_S$ and $\mathbf{source}_T$ being identity functors $\mathbf{id}_A$: the only possible choice for θ is also $\mathbf{id}_A$, which forces $F = G$.

From now on, whenever we consider **Cat** as a 2-category (and thus bicategory), we do mean $\mathbf{Cat}^{\Rightarrow}$, unless specified otherwise.

The free symmetric monoidal category construction.

As we will see below, the interpretation of the exponential modality $A \mapsto !A$ in the bicategories $\mathcal{W} = \mathbf{Dist}$ and $\mathcal{W} = \mathbf{Span}$ is derived in [13,27] from the 2-monad $\mathbf{Sym} : \mathbf{Cat} \to \mathbf{Cat}$ which transports every small category A to the symmetric strict monoidal category $\mathbf{Sym}\,A$ freely generated by A. In both cases, **Sym** categorifies the finite multiset monad $\mathbf{MSet}_{\mathrm{fin}} : \mathbf{Set} \to \mathbf{Set}$ from which one derives the exponential modality $A \mapsto !A$ in the relational semantics of linear logic.

We recall that the category $\mathbf{Sym}\,A$ has its objects defined as finite words $\langle a_1, \ldots, a_n \rangle$ of objects of A, and its morphisms $\langle a_1, \ldots, a_n \rangle \to \langle a'_1, \ldots, a'_n \rangle$ defined as tuples $\langle \sigma; f_1, \ldots, f_n \rangle$ consisting of a permutation $\sigma \in \mathfrak{S}_n$ together with a finite word $\langle f_1, \ldots, f_n \rangle$ of morphisms $f_i : a_{\sigma(i)} \to a'_i$ in the category A ; there are no morphisms between words of different lengths. The construction defines a 2-monad $\mathbf{Sym} : \mathbf{Cat} \to \mathbf{Cat}$ equipped with cartesian natural 2-transformations: the multiplication $\mu : \mathbf{Sym} \circ \mathbf{Sym} \Rightarrow \mathbf{Sym}$ given by the concatenation of words of words, and unit $\eta : Id \Rightarrow \mathbf{Sym}$ given by the empty word.

Moreover, every pair of small categories A and B induces a pair of functors

$$\mathbf{Sym}\,(A+B) \underset{\mathbf{concat}_{A,B}}{\overset{\mathbf{deshuffle}_{A,B}}{\rightleftarrows}} \mathbf{Sym}\,A \times \mathbf{Sym}\,B \tag{3}$$

defining an equivalence of categories, as noted in [13, Lemma 5.2.1]. These functors, which can be derived from the universal property of **Sym**, are explicitly defined as follows:

$$\begin{array}{rcccc} \mathbf{concat}_{A,B} & = & (\langle a_1,\ldots,a_m\rangle, \langle b_1,\ldots,b_n\rangle) & \longmapsto & \langle a_1,\ldots,a_m,b_1,\ldots,b_n\rangle \\ \mathbf{deshuffle}_{A,B} & = & \langle c_1,\ldots,c_{m+n}\rangle & \longmapsto & (\langle a_1,\ldots,a_m\rangle, \langle b_1,\ldots,b_n\rangle) \end{array}$$

where the finite words $\langle a_1, \ldots, a_m \rangle$ and $\langle b_1, \ldots, b_n \rangle$ are uniquely characterized by the fact that there exist jointly surjective functions $f : \{1, \ldots, m\} \to \{1, \ldots, m+n\}$ and $g : \{1, \ldots, n\} \to \{1, \ldots, m+n\}$ such that $c_{f(i)} = a_i$ for all $1 \leq i \leq m$ and $c_{g(j)} = b_j$ for all $1 \leq j \leq n$. The families **concat** and **deshuffle** define natural transformations between the 2-functors $\mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}$ defined as $A, B \mapsto \mathbf{Sym}\,(A+B)$ and $A, B \mapsto \mathbf{Sym}\,A \times \mathbf{Sym}\,B$. Both natural 2-transformations are moreover cartesian, in the sense that the commuting squares for naturality are pullback squares. The 2-monad $\mathbf{Sym} : \mathbf{Cat} \to \mathbf{Cat}$ also comes with functors

$$\mathbf{Sym}\,\mathbf{0} \underset{\mathbf{concat}_{\mathbf{0}}}{\overset{\mathbf{deshuffle}_{\mathbf{0}}}{\rightleftarrows}} \mathbf{1} \tag{4}$$

defining an equivalence of categories, where $\mathbf{0}$ is the empty category, and $\mathbf{1}$ is the singleton category. The family of functors **concat** equips the 2-functor $\mathbf{Sym} : \mathbf{Cat} \to \mathbf{Cat}$ with the structure of a symmetric monoidal 2-functor from $(\mathbf{Cat}, +, \mathbf{0})$ to $(\mathbf{Cat}, \times, \mathbf{1})$, whose monoidal coercions, in both directions, are categorical equivalences. For these reasons the 2-monad **Sym** may be called a cartesian symmetric monoidal 2-monad.

From now on, we will call (3) and (4) *Seely equivalences* since they induce what plays the rôle of *Seely isomorphisms* in the categorified interpretation of linear logic discussed below.

The distributor model of linear logic.

Fiore, Gambino, Hyland and Winskel explained in their seminal and celebrated paper on generalised species [13] how to use the free symmetric monoidal category 2-monad $\mathbf{Sym} : \mathbf{Cat} \to \mathbf{Cat}$ in order to turn

Dist into a 2-dimensional model of linear logic. The construction relies on two main ingredients.

The first ingredient is the existence of a distributivity law between **Sym** and the presheaf (relative) monad on **Cat** [14], which enables one to extend the 2-monad $\mathbf{Sym} : \mathbf{Cat} \to \mathbf{Cat}$ to a weak 2-monad $\mathbf{Sym}_? : \mathbf{Dist} \to \mathbf{Dist}$ – see the discussion in [24]. The weak 2-monad is then transformed by duality into a weak 2-comonad $\mathbf{Sym}_! : \mathbf{Dist} \to \mathbf{Dist}$ obtained by precomposing and postcomposing $\mathbf{Sym}_?$ with the 2-functor $\mathbf{op} : \mathbf{Dist} \to \mathbf{Dist}^{op(1)}$ which sends every small category A to its opposite category A^{op}. Here, we write $\mathbf{Dist}^{op(1)}$ for the bicategory **Dist** where the orientation of 1-cells (but not of 2-cells) has been reversed.

The second ingredient, less known but just as important as the first one, is the fact that, as discussed above, the weak 2-functor $L : \mathbf{Cat}^{=} \to \mathbf{Dist}$ does extend to a weak 2-functor $L : \mathbf{Cat}^{\Rightarrow} \to \mathbf{Dist}$. This ensures that L transports the Seely equivalence (3) and (4) from the 2-category **Cat** to the corresponding Seely equivalence in the bicategory **Dist**.

An apparent obstruction to the construction of the span model of linear logic.

Somewhat surprisingly, one bumps against a serious obstruction when one tries to obtain a model of linear logic based on spans, by extending the 2-monad $\mathbf{Sym} : \mathbf{Cat} \to \mathbf{Cat}$ to the bicategory **Span** instead of **Dist**. Indeed, as mentioned above and in contrast to what happens with **Dist**, the weak 2-functor $L : \mathbf{Cat}^{=} \to \mathbf{Span}$ does not extend to $\mathbf{Cat}^{\Rightarrow}$. Hence, it does not transport the Seely equivalence (3) living in the 2-category $\mathbf{Cat}^{\Rightarrow}$ to a Seely equivalence living in the bicategory **Span**. On the other hand, we are not far from a linear logic model, since **Sym** lifts to a weak 2-monad on **Span** as a cartesian 1-monad on **Cat** [4,30]; **Sym** is moreover both lax and oplax monoidal on **Cat**, as supported by the Seely equivalences (3). For this reason, **Sym** may be called a bilax monoidal (2-)monad on **Cat** [1], and the whole structure lifts to **Span** because (3) is cartesian. However, the fact that L does not extend to the 2-dimensional structure of $\mathbf{Cat}^{\Rightarrow}$ implies in particular that the equivalence (3) in $\mathbf{Cat}^{\Rightarrow}$ is *not* transported to an equivalence in the bicategory **Span**.

Melliès [27] explained how to use abstract homotopy theory in order to resolve the obstruction and to define a homotopy variant **HoSpan** of the original bicategory **Span** suitable for the interpretation of linear logic. The philosophy of abstract homotopy theory is that the "good" homotopic constructions performed in a Quillen model category should not depend on the choice of objects up to homotopy. Given an object, there is often a large number of alternative choices of objects equivalent up to homotopy, and a good practice is thus to pick a specific object defined in a principled way.

In his original construction, Melliès [27] interpreted proofs using fibrant objects and fibrant replacements in the hom-category $\mathbf{Span}(A, B)$. Our goal in the present work is to give a more direct and explicit construction of this 2-dimensional model, where the fibrant spans are replaced by separately fibrant spans carefully chosen at each step of the interpretation. This construction highlights the combinatorial nature of the interpretation, and reveals the existence of a factorization system on shuffles in the symmetric monoidal category $\mathbf{Sym}(A)$, seen as an operad.

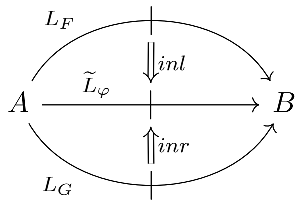
Plan of the paper.

After this long but necessary introduction on the bicategorical semantics of linear logic, we will expose in more detail the homotopy span model recently formulated by Melliès [27] in section 2. In particular, we explain how it takes advantage of homotopy ideas in order to transport the Seely equivalences in $\mathbf{Cat}^{\Rightarrow}$ to Seely equivalences in the homotopy variant **HoSpan**, constructed from **Span** using the natural Quillen model structure on **Cat**.

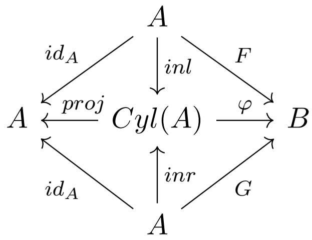
In section 3 and onwards, we describe how the formulae and proofs of linear logic are interpreted in our new and more explicit formulation of the model, based on shuffles. The interpretation relies in particular on a factorization property of the operad $\mathbf{Sym}(A)$, of independent interest, which we chose to present in section 4. One main contribution is to show how this factorization property allows to give a simple and concrete separately fibrant interpretation of promotion: this is done in section 5. We then conclude in section 6.

2 The homotopy span model of linear logic

The key observation made in [27] is that a natural isomorphism $\varphi : F \Rightarrow G$ between functors $F, G : A \to B$ in **Cat** is *not* transported to an invertible 2-cell $L_\varphi : L_F \Rightarrow L_G$ in the bicategory **Span**, but to a *cospan* of 2-cells from L_F and L_G to the span $\widetilde{L}_\varphi$:



defined as follows



Here, the span $\widetilde{L}_\varphi$ is defined as $(Cyl(A), proj, \varphi)$ where $Cyl(A)$ denotes the *cylinder category* defined as the cartesian product $Cyl(A) = \mathbb{J} \times A$ of the small category A with the *interval groupoid* $\mathbb{J}$ with two objects 0 and 1 and a unique isomorphism $0 \to 1$ between them. The functors *inl*, *inr* and *proj* are defined as $inl(a) := (0, a)$, $inr(a) := (1, a)$ and $proj(i, a) := a$ for $i \in \{0, 1\}$, while the functor $\varphi : \mathbb{J} \times A \to B$ provides an alternative description of the natural isomorphism φ with $\varphi(0, a) = Fa$, $\varphi(1, a) = Ga$, $\varphi(0 \to 1, a) = \varphi_a$.

An additional observation made in [27] is that the functors $inl, inr : A \to Cyl(A)$ are fully faithful and essentially surjective, and thus equivalences of categories. This strongly suggests to *localize* (or formally invert) in the hom-category **Span**(A, B) every morphism given by a simulation 2-cell (2) whose underlying functor $\theta : S \to T$ is an equivalence of categories. This localization can be neatly performed using the tools of abstract homotopy theory, and in particular the notion of Quillen model category. Recall [25,31] that the category **Cat** comes equipped the natural Quillen structure $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ defined as follows:

- the class $\mathcal{W}$ of *weak equivalence* is the class of equivalences of categories,
- the class $\mathcal{C}$ of *cofibrations* is the class of functors injective on objects,
- the class $\mathcal{F}$ of *fibrations* is the class of isofibrations – recall that a functor $F : A \to B$ is an isofibration if for any object $a \in A$ and any isomorphism g with target $F(a)$ in B, there exists an isomorphism f with target a in A such that $g = F(f)$.

From this follows [21] that every hom-category **Span**$(A, B) =$ **Cat**$/A \times B$ comes equipped as a slice category with a Quillen model structure $(\mathcal{W}_{A,B}, \mathcal{C}_{A,B}, \mathcal{F}_{A,B})$:

- a weak equivalence in $\mathcal{W}_{A,B}$ is a simulation 2-cell $\theta : \mathcal{S} \Rightarrow \mathcal{T}$ whose underlying functor $\theta : S \to T$ is an element of $\mathcal{W}$, that is, an equivalence of categories;
- a cofibration in $\mathcal{C}_{A,B}$ is a simulation 2-cell $\theta : \mathcal{S} \Rightarrow \mathcal{T}$ whose underlying functor $\theta : S \to T$ is an element of $\mathcal{C}$, that is, a functor injective on objects;
- a fibration in $\mathcal{F}_{A,B}$ is a simulation 2-cell $\theta : \mathcal{S} \Rightarrow \mathcal{T}$ whose underlying functor $\theta : S \to T$ is an element of $\mathcal{F}$, that is, a isofibration.

This defines what we like to call the natural model structure on **Span**$(A, B) =$ **Cat**$/A \times B$. The category obtained by localizing in **Span**(A, B) all weak equivalences in $\mathcal{W}_{A,B}$ is then the homotopy category **HoSpan**$(A, B) =$ **Span**$(A, B)[\mathcal{W}_{A,B}^{-1}]$ associated to this natural Quillen model structure. One important point is that this category **HoSpan**(A, B) cannot be easily described by applying a purely 2-categorical approach: note in particular that although the weak equivalences $inl, inr : A \to Cyl(A)$ are adjoint equivalences in the 2-category **Cat**, they are not adjoint equivalences in the 2-category **Span**(A, B) of functorial spans over A and B.

Morphisms in a Quillen model category.

We follow a well-established tradition in homotopy theory and draw every cofibration $c : A \rightarrowtail B$ and every fibration $f : A \twoheadrightarrow B$. We also indicate with a sign $\sim$ that a morphism $w : A \to B$ is a weak equivalence. Typically, an *acyclic cofibration* $v : A \overset{\sim}{\rightarrowtail} B$ is defined as an element of $\mathcal{W} \cap \mathcal{C}$, that is, a weak equivalence which is at the same time a cofibration; and an *acyclic fibration* $w : A \overset{\sim}{\twoheadrightarrow} B$ is an element of $\mathcal{W} \cap \mathcal{F}$. By way of illustration, we could rewrite the Seely equivalence (3) as

$$\mathbf{deshuffle}_{A,B} \;:\; \mathbf{Sym}\,(A+B) \;\underset{\sim}{\overset{\sim}{\rightleftarrows}}\; \mathbf{Sym}\,A \times \mathbf{Sym}\,B \;:\; \mathbf{concat}_{A,B} \tag{5}$$

in order to stress the fact that $\mathbf{concat}_{A,B}$ is an acyclic cofibration and that $\mathbf{deshuffle}_{A,B}$ is an acyclic fibration in the natural Quillen model structure on **Cat**. Indeed, we have already stated that $\mathbf{concat}_{A,B}$ and $\mathbf{deshuffle}_{A,B}$ form an equivalence of categories, and the injectivity of $\mathbf{concat}_{A,B}$ on objects is obvious. The fact that $\mathbf{deshuffle}_{A,B}$ is an isofibration can be deduced from the fact that $\mathbf{deshuffle}_{\mathbf{1},\mathbf{1}}$ is an isofibration for purely combinatorial reasons, and the fact that isofibrations are preserved by pullbacks – here we also apply the fact that **deshuffle** is a cartesian natural 2-transformation, to the specific case of the unique functors $F : A \to \mathbf{1}$ and $G : B \to \mathbf{1}$ to the terminal category.

An object A is called *cofibrant* when the unique morphism $0 \to A$ from the initial object 0 is a cofibration, and *fibrant* when the unique morphism $A \to 1$ to the terminal object 1 is a fibration. In a Quillen model category, every morphism $f : A \to B$ can be factored (in a non necessarily unique way) as an acyclic cofibration followed by a fibration $f : A \overset{\sim}{\rightarrowtail} A' \twoheadrightarrow B$, and as a cofibration followed by an acyclic fibration $f : A \rightarrowtail B' \overset{\sim}{\twoheadrightarrow} B$. When $B = 1$ and $f : A \to 1$ is the unique morphism to the terminal object, any object A' involved in a factorization of the first form is called a *fibrant replacement* of A. Note that an object A may have several (non necessarily isomorphic) fibrant replacements.

The homotopy category $\mathbf{HoSpan}(A, B)$ of spans.

One advantage of applying the ideas of abstract homotopy theory is that it provides a nice description of the homotopy category $\mathbf{Ho}\,C = C[\mathcal{W}^{-1}]$ obtained by localizing (= formally inverting) all the weak equivalences of a Quillen model category C with class $\mathcal{W}$ of weak equivalences, class $\mathcal{C}$ of cofibrations and class $\mathcal{F}$ of fibrations. The homotopy category $\mathbf{Ho}\,C = C[\mathcal{W}^{-1}]$ is the category of fibrant and cofibrant objects in C, whose morphisms are defined as the morphisms of the original category $\mathbf{Span}(A, B)$ considered modulo a specific homotopy relation, see [23,21] for details. In the particular case of the homotopy category $\mathbf{HoSpan}(A, B) := \mathbf{Span}(A, B)[\mathcal{W}_{A,B}^{-1}]$ the characterization of the fibrant and cofibrant objects of $\mathbf{Span}(A, B)$ goes as follows. Every small category in **Cat** is both fibrant and cofibrant. The situation is a bit different in the general case of the natural Quillen model structure on $\mathbf{Span}(A, B) = \mathbf{Cat}/A \times B$. In that case, every span $\mathcal{S} = (S, \mathbf{source}, \mathbf{target})$ is cofibrant, because the initial object $0_{A,B}$ in $\mathbf{Span}(A, B)$ has source 0 and thus the unique morphism $0_{A,B} \to \mathcal{S}$ is injective on objects. On the other hand, a span is fibrant precisely when its canonical pairing functor

$$\langle \mathbf{source}, \mathbf{target} \rangle \quad : \quad S \longrightarrow\!\!\!\!\!\rightarrow A \times B \tag{6}$$

is an isofibration, because the terminal object $1_{A,B}$ in $\mathbf{Span}(A, B)$ is the identity functor $A \times B \to A \times B$ and the unique morphism $\mathcal{S} \to 1_{A,B}$ coincides with (6).

The category **HoSpan** can be thus described as follows. Its objects are the spans $\mathcal{S} = (S, \mathbf{source}, \mathbf{target})$ whose pairing functor (6) is an isofibration, and its morphisms are the simulation 2-cells (2) considered up to the following homotopy relation: $\varphi, \psi : \mathcal{S} \Rightarrow \mathcal{T}$ are declared *homotopic up to a natural isomorphism*, and we write $\varphi \sim_{A,B} \psi$, when there exists a natural isomorphism $\alpha : \varphi \Rightarrow \psi$ between the underlying functors $\varphi, \psi : S \to T$ of the simulation 2-cells, which is *vertical* in $\mathbf{Cat}/A \times B$, in the expected sense that

- the natural isomorphism $\mathbf{source}_T \circ \alpha$ is the identity on $\mathbf{source}_S$,
- the natural isomorphism $\mathbf{target}_T \circ \alpha$ is the identity on $\mathbf{target}_S$.

The fact that every small category is fibrant in the small category **Cat** ensures that the composition $\mathcal{S} \circ \mathcal{R}$

of spans $\mathcal{R} \in \mathbf{Span}(A, B)$ and $\mathcal{S} \in \mathbf{Span}(B, C)$, given by the pullback

$$\begin{array}{ccccccccc} & & & & R \times_B S & & & & \\ & & & \overset{\pi_1}{\swarrow\!\!\swarrow} & & \overset{\pi_2}{\searrow\!\!\searrow} & & & \\ & & R & & \text{pullback} & & S & & \\ & \overset{\mathbf{source}_R}{\swarrow\!\!\swarrow} & & \underset{\mathbf{target}_R}{\searrow\!\!\searrow} & & \underset{\mathbf{source}_S}{\swarrow\!\!\swarrow} & & \overset{\mathbf{target}_S}{\searrow\!\!\searrow} & \\ A & & & & B & & & & C \end{array} \tag{7}$$

in **Cat**, is fibrant as soon as $\mathcal{R}$ and $\mathcal{S}$ are. The homotopy relations between simulation 2-cells are moreover preserved. In this way, one obtains a composition functor

$$\circ_{A,B,C} \quad : \quad \mathbf{HoSpan}(B, C) \times \mathbf{HoSpan}(A, B) \longrightarrow \mathbf{HoSpan}(A, C)$$

equipped with the appropriate associator and unitors required to turn the family of homotopy categories $\mathbf{HoSpan}(A, B)$ into a bicategory **HoSpan**. The homotopy construction has the beneficial effect of resolving the obstruction discussed above, and to define a homotopy variant **HoSpan** of the bicategory **Span** equipped with a (weak) 2-functor

$$\mathbf{Cat}^{\simeq} \longrightarrow \mathbf{HoSpan}$$

from the 2-category $\mathbf{Cat}^{\simeq}$ obtained by restricting the 2-category $\mathbf{Cat}^{\Rightarrow}$ to its invertible 2-cells, *i.e.* to natural isomorphisms. This is sufficient to transport to the bicategory **HoSpan** the Seely equivalences living in **Cat**. Following this approach, Melliès obtains a denotational model of linear logic where every formula A is interpreted as a small category $[\![A]\!]$ and where every derivation tree $\pi \vdash A_1\,,\, A_2\,,\,\ldots\,,\, A_n$ is interpreted as an isofibration

$$[\![\pi]\!] \longrightarrow\!\!\!\!\!\rightarrow [\![A_1]\!] \times \cdots \times [\![A_n]\!]. \tag{8}$$

Moreover, the inherently homotopic nature of the construction of **HoSpan** means that the interpretations of proofs $[\![\pi]\!]$ provides a *homotopy invariant* with respect to cut-elimination. In other words, two proofs π and π' equal modulo cut-elimination have isomorphic interpretations $[\![\pi]\!]$ and $[\![\pi']\!]$ in the bicategory **HoSpan**, which means that their interpretations are related by equivalences of categories (the notion of homotopy in the natural Quillen model structure) instead of categorical isomorphisms, see [27] for a discussion.

Canonical fibrant replacements.

We present the construction of a canonical fibrant replacement of any functor. Given $B \xrightarrow{F} A$, we define the *canonical fibrant replacement* of F as follows. First, we define a category $\mathsf{Iso}(B)$, whose objects consist of pairs of one object b of B together with an isomorphism $\alpha : a \to F(b)$ in A. A morphism from (b, α) to (b', α') consists of a morphism $\beta : b \to b'$ and a morphism $\gamma : a \to a'$ making the following diagram commute

$$\begin{array}{ccc} a & \xrightarrow{\gamma} & a' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ F(b) & \xrightarrow{F(\beta)} & F(b') \end{array} .$$

Note that such a morphism (β, γ) is in fact uniquely determined by β because α and α' are isomorphisms. Then, we define

$$\mathsf{Iso}(B) \xrightarrow{\mathsf{Iso}(F)}\!\!\!\!\!\rightarrow A$$

by the maps

$$\begin{array}{ccc} (b, \alpha : a \to F(b)) & \mapsto & a \\ (\beta : b \to b', \gamma : a \to a') & \mapsto & \gamma\,. \end{array}$$

The functor $\mathsf{Iso}(F)$ is indeed an isofibration: given any isomorphism $g : a' \to a = \mathsf{Iso}(F)(b, \alpha)$ we immediately obtain $\mathsf{Iso}(F)(\mathbf{id}_b, g) = g$. We moreover obtain $B \overset{\sim}{\rightarrowtail} \mathsf{Iso}(B)$ by the inclusion functor sending each object b to the pair $(b, \mathbf{id}_{F(b)})$: it is obviously injective of objects and essentially surjective; it is also fully faithful, because a morphism (β, γ) in $\mathsf{Iso}(B)$ is uniquely determined by β.

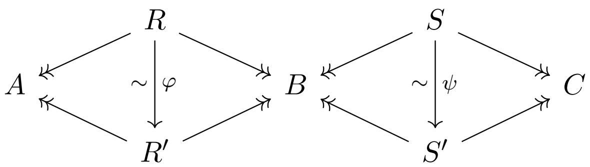
An explicit shuffle construction of the span homotopy model.

The original presentation of the span homotopy model of linear logic by Melliès [27] relied extensively on fibrant objects and fibrant replacements in $\mathbf{Span}(A, B)$. Here, we follow alternative recipe to construct an equivalent, but more direct and explicit interpretation of linear logic.

We declare that a span $\mathcal{S}$ is *separately fibrant* when the two morphisms **source** and **target** are isofibrations, that is, fibrations in the natural Quillen model structure of **Cat**. Note that every span fibrant in $\mathbf{Span}(A, B)$ is separately fibrant: this is a formal consequence of the fact that every small category is fibrant in **Cat**. On the other hand, the notion of separately fibrant span is strictly more permissive: the diagonal $\Delta : \mathbb{J} \to \mathbb{J} \times \mathbb{J}$ is obviously separately fibrant (each component is the identity on $\mathbb{J}$), whereas it is not fibrant.

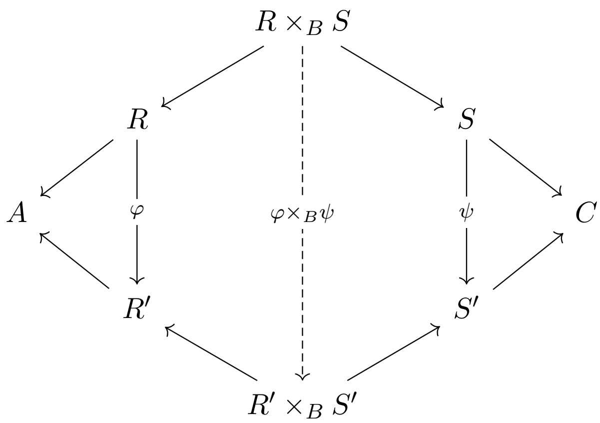
The full subcategory of separately fibrant spans in $\mathbf{Span}(A, B)$ is noted $\mathbf{SFibSpan}(A, B)$. Note that separately fibrant spans compose horizontally as separately fibrant spans, and thus define a sub-bicategory **SFibSpan** of the bicategory **Span**. The key idea of our explicit construction is to exploit the following property:

Proposition 2.1 *Suppose that $\varphi : \mathcal{R} \Rightarrow \mathcal{R}'$ and $\psi : \mathcal{S} \Rightarrow \mathcal{S}'$ are weak equivalences between separately fibrant spans*



In that case, the composite 2-cell $\psi \circ \varphi : \mathcal{S} \circ \mathcal{R} \Rightarrow \mathcal{S}' \circ \mathcal{R}'$ is a weak equivalence.

Proof. We first recall that the horizontal composite $\psi \circ \varphi$ is defined exploiting the universal property of the pullback $R' \times_B S'$:



Hence, for $(r, s) \in R \times_B S$, we have that $\varphi \times_B \psi$ is defined pointwise $(\varphi \times_B \psi)(r, s) = (\varphi(s), \psi(r))$. Since the pullback categories are subcategories of the cartesian product, and φ and ψ are separately weak equivalences, it follows that that $\varphi \times_B \psi$ is fully-faithful. Since $\mathcal{R}$ and $\mathcal{S}$ are separately fibrant and pullbacks preserve isofibrations, we can infer that $\varphi \times_B \psi$ is also essentially surjective. □

The result means that the composition of separately fibrant spans – that is, the horizontal composition in **SFibSpan** – preserves weak equivalences on the left and on the right. For that reason, it is convenient to describe the bicategory **HoSpan** as the bicategory **SFibSpan** where every hom-category $\mathbf{SFibSpan}(A, B)$ is localized with respect to weak equivalences. Note that two weak equivalences φ and ψ do not necessarily

$$A, B ::= 0 \mid \top \mid 1 \mid \bot \mid A \oplus B \mid A \,\&\, B \mid A \otimes B \mid A \parr B \mid\, !\,A \mid\, ?\,A \mid X \mid X^{\bot}$$

$$\frac{}{\vdash A^{\bot}, A}\ \text{axiom} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\bot}}{\vdash \Gamma, \Delta}\ \text{cut} \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\ \text{exch}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot$$

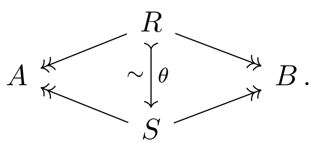
$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \oplus A_2}\ \oplus_i \qquad \frac{}{\vdash \Gamma, \top}\ \top$$

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ? \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ \text{contr} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ \text{weak}$$

Fig. 1. Formulae and rules of Linear Logic.

compose horizontally as a weak equivalence $\psi \circ \varphi$ in the bicategory **Span** when the spans $\mathcal{R}$, $\mathcal{R}'$, $\mathcal{S}$, $\mathcal{S}'$ are not separately fibrant.

This observation means that we can perform homotopic constructions in the bicategory **HoSpan** by taking general spans in $\mathbf{Span}(A, B)$ and by replacing them by their *separately fibrant replacements* in **SFibSpan** instead of their fibrant replacements in **HoSpan**. Here, we define a *separately fibrant replacement* of a span $\mathcal{R}$ from A to B as a separately fibrant span $\mathcal{S}$ together with an acyclic cofibration $\theta : \mathcal{R} \Rightarrow \mathcal{S}$, as depicted below:



Our main contribution in the present paper is to show that the two canonical functors

$$\otimes \quad : \quad \mathbf{Sym}\, A \times \mathbf{Sym}\, A \longrightarrow \mathbf{Sym}\, A$$

$$\mathbf{dist}_{\mathbf{Sym}B,A} \quad : \quad \mathbf{Sym}\,(\mathbf{Sym}\, B \times A) \longrightarrow \mathbf{Sym}\, B \times \mathbf{Sym} A$$

underlying the *contraction rule* and the *promotion rule* of linear logic have very simple separately fibrant replacements, while their fibrant replacements are more complicated.

We find convenient to introduce the following notations. Given small category X and a list $\Gamma = A_1, \ldots, A_n$ of small categories, we denote by $\boldsymbol{f} : X \longrightarrow\!\!\!\!\circ\ \Gamma$ any n-fold span: that is given by a family of functors $f_i : X \longrightarrow A_i$ for $1 \leq i \leq n$ – this induces a functor to the product $\prod \Gamma = A_1 \times \cdots \times A_n$ by tupling $\langle f_1, \ldots, f_n \rangle : X \longrightarrow \prod \Gamma$. We generalize the notion of separately fibrant spans to such n-fold spans: we say $\boldsymbol{f}$ is separately fibrant, and we write $\boldsymbol{f} : X \multimap\!\!\!\rightarrow \Gamma$, when each f_i is a fibration.

Now, instead of a fibration $[\![\pi]\!] \longrightarrow\!\!\!\!\rightarrow [\![A_1]\!] \times \cdots \times [\![A_n]\!]$, we will interpret a proof $\pi \vdash \Gamma = A_1, \ldots, A_n$ as a separately fibrant n-fold span:

$$[\![\pi]\!] \multimap\!\!\!\rightarrow [\![\Gamma]\!] = [\![A_1]\!], \ldots, [\![A_n]\!] \quad . \tag{9}$$

Note that one can always recover a fibration from a proof: setting $A = A_1 \parr \cdots \parr A_n$ and $\pi^{\parr} \vdash A$ the proof obtained from π by iterating the ($\parr$) rule, the separately fibrant span $[\![\pi^{\parr}]\!]$ consists of a single fibration:

$$[\![\pi^{\parr}]\!] \longrightarrow\!\!\!\!\rightarrow [\![A]\!] \quad . \tag{10}$$

In a sense, distinguishing between (9) and (10) is precisely what allows us to provide a very simple and concrete interpretation of proofs.

3 Interpretation of linear logic

We explain how to interpret the formulae and proofs of classical linear logic which we describe in the usual syntactic way using a (one sided) sequent calculus, that we recall in fig. 1. We suppose given an interpretation of each atomic formula X of linear logic as a small category $[\![X]\!] \in \mathbf{Cat}$. The interpretation is then performed in two stages. First, the formulae of classical linear logic are interpreted by structural induction:

$$[\![0]\!] = [\![\top]\!] = \mathbf{0} \qquad [\![A \oplus B]\!] = [\![A \,\&\, B]\!] = [\![A]\!] + [\![B]\!] \qquad [\![X^\perp]\!] = [\![X]\!]$$
$$[\![1]\!] = [\![\perp]\!] = \mathbf{1} \qquad [\![A \otimes B]\!] = [\![A \parr B]\!] = [\![A]\!] \times [\![B]\!] \qquad [\![!\, A]\!] = [\![?\, A]\!] = \mathbf{Sym}[\![A]\!]$$

so that each sequent $\Gamma = A_1, \ldots, A_n$ induces a tuple of small categories which we simply write $[\![\Gamma]\!] = [\![A_1]\!], \ldots, [\![A_n]\!]$. The interpretation of the derivation tree π is defined by structural induction, by inspecting each rule of the sequent calculus. Note that the interpretation of the exchange rule (exch) is straightforward: we let the same permutation act on the tuple of isofibrations. From now on, we find more convenient and easier for the reader to drop the semantic bracket $[\![-]\!]$ from our notations.

Axiom.
The axiom rule is simply interpreted as the identity span:

$$A \xleftarrow{\;id\;} A \xrightarrow{\;id\;} A \tag{11}$$

It is important to note that although this span is separately fibrant because each of its component is an isofibration, it is in general not fibrant in the model category $\mathbf{HoSpan}(A, A)$ because the diagonal functor $A \to A \times A$ itself is not an isofibration in general. This interpretation thus substantially improves the original model [27] where the axiom rule was interpreted as the fibrant replacement of the diagonal defined by the category of isomorphisms of A, equipped with is source and target functors.

Cut.
Consider two proofs $\pi_1 \vdash \Gamma\,, A$ and $\pi_2 \vdash \Delta\,, A^\perp$ interpreted as separately fibrant spans

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi_1 \twoheadrightarrow A \qquad\qquad A \twoheadleftarrow \pi_2 -\!\!\circ\!\!\twoheadrightarrow \Delta$$

The interpretation of the derivation tree π obtained by cutting π_1 and π_2, is obtained by computing the pullback $\pi := \pi_1 \times_A \pi_2$ of the cospan $\pi_1 \twoheadrightarrow A \twoheadleftarrow \pi_2$ consisting of the two isofibrations on the cut formula. Isofibrations are closed under pullbacks, and this thus yields a separately fibrant span consisting of the two projection maps:

$$\pi_1 \xtwoheadleftarrow{\;\texttt{fst}\;} \pi_1 \times_A \pi_2 \xtwoheadrightarrow{\;\texttt{snd}\;} \pi_2$$

The desired interpretation is then obtained

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi_1 \xtwoheadleftarrow{\;\texttt{fst}\;} \pi_1 \times_A \pi_2 \xtwoheadrightarrow{\;\texttt{snd}\;} \pi_2 -\!\!\circ\!\!\twoheadrightarrow \Delta$$

by postcomposing the projection maps with the components of the separately fibrant spans $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$.

Additives
The interpretation of additives is easy and does not require any explicit fibrant replacement. Consider two proofs $\pi_1 \vdash \Gamma\,, A$ and $\pi_2 \vdash \Gamma\,, B$ interpreted as separately fibrant spans

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi_1 \twoheadrightarrow A \qquad\qquad \Gamma \twoheadleftarrow\!\!\circ\!\!- \pi_2 \twoheadrightarrow B$$

The interpretation of the derivation tree obtained by combining π_1 and π_2 using a with rule (&) is defined as the sum of the two interpretations

$$\Gamma \twoheadleftarrow\!\!\circ\!\!\longrightarrow\!\!\!\!\!\!\!\longleftarrow \ \pi_1 + \pi_2 \longrightarrow\!\!\!\!\!\twoheadrightarrow A + B$$

where the context maps from $\pi_1 + \pi_2$ to Γ are defined by the universality property of the sum, while the map from $\pi_1 + \pi_2$ to $A + B$ is defined as the image of the two maps from π_1 to A and from π_2 to B by the sum functor. The universal construction as well as the sum functor preserve isofibrations, and the resulting span is thus separately fibrant, as indicated in the diagram.

Similarly, the proof reduced to a $(\top)$ rule is interpreted as the separately fibrant span

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \ \mathbf{0} \twoheadrightarrow \mathbf{0}$$

where the context isofibrations are defined by the universality property of the initial category $\mathbf{0}$ and where the isofibration from $\mathbf{0}$ to $\mathbf{0}$ is the identity.

On the other hand, the plus rules $(\oplus_1)$ and $(\oplus_2)$ are interpreted by leaving the context maps unchanged and postcomposing the map to the interpretation of the principal formula A or B with the left and right injections, which are isofibrations:

$$A \twoheadrightarrow A + B \qquad \text{and} \qquad B \twoheadrightarrow A + B\,.$$

Multiplicatives (tensor, one).

The interpretation is easy and does not require any explicit fibrant replacement. Consider two proofs $\pi_1 \vdash \Gamma\,, A$ and $\pi_2 \vdash \Delta\,, B$ interpreted as separately fibrant spans

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \ \pi_1 \twoheadrightarrow A \qquad\qquad \Delta \twoheadleftarrow\!\!\circ\!\!- \ \pi_2 \twoheadrightarrow B$$

The interpretation of the derivation tree π obtained by combining π_1 and π_2 using a tensor rule $(\otimes)$ is defined as the cartesian product $\pi := \pi_1 \times \pi_2$ of the two interpretations with the separately fibrant span

$$\begin{array}{ccccc} \Gamma \twoheadleftarrow\!\!\circ\!\!- & \pi_1 & \overset{\texttt{fst}}{\twoheadleftarrow} & & \\ & & & \pi_1 \times \pi_2 & \twoheadrightarrow A \times B \\ \Delta \twoheadleftarrow\!\!\circ\!\!- & \pi_2 & \underset{\texttt{snd}}{\twoheadleftarrow} & & \end{array}$$

obtained by postcomposing the projection maps with the components of the separately fibrant spans $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$. Here, we use a general property of model categories that the class $\mathcal{F}$ of fibrations (= the isofibrations in the case of the natural Quillen model structure on **Cat**) is closed under cartesian product. We also use the fact that every object is fibrant, and that the projection maps `fst` and `snd` are thus isofibrations. The proof reduced to the rule (1) introducing the tensor unit is interpreted by the identity map $\mathbf{1} \to \mathbf{1}$ which defines an isofibration on the category $\mathbf{1}$.

Multiplicatives (par, bot).

One interesting aspect of our presentation of the homotopy span model is that the interpretation of the par rule $(⅋)$ is nontrivial and involves a fibrant replacement. Consider a proof $\pi \vdash \Gamma\,, A\,, B$ interpreted as a separately fibrant span

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \ \pi \begin{array}{l} \twoheadrightarrow A \\ \twoheadrightarrow B \end{array}$$

The functor $\pi \to A \times B$ obtained by the universality property of the cartesian product in **Cat** is not necessarily an isofibration, but it can be factored in the Quillen model category as an acyclic cofibration followed by an isofibration:

$$\pi \overset{\sim}{\rightarrowtail} \mathbf{Iso}(\pi) \twoheadrightarrow A \times B$$

where $\mathbf{Iso}(\pi) \to A \times B$ is defined as the canonical fibrant replacement of $\pi \to A \times B$. Note that the category π' comes with an acyclic fibration

$$\pi \overset{\sim}{\twoheadleftarrow} \mathbf{Iso}(\pi) \twoheadrightarrow A \times B$$

The interpretation of the derivation tree $\pi' \vdash \Gamma\,,\, A \parr B$ obtained by applying the par ($\parr$) rule is then obtained by taking the category π' as support and postcomposing

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi \overset{\sim}{\twoheadleftarrow} \mathbf{Iso}(\pi) \twoheadrightarrow A \times B$$

with the context maps of the original separately fibrant span.

On the other hand, the interpretation of the par unit ($\bot$) is easy to define and does not require any explicit fibrant replacement. Given a proof $\pi \vdash \Gamma$ interpreted as the separately fibrant span

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi$$

the interpretation of the proof $\pi' \vdash \Gamma\,,\,\bot$ obtained after the rule ($\bot$) is simply defined by adding a functor to the terminal category $\mathbf{1}$ in the following way:

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi \twoheadrightarrow \mathbf{1}$$

Here, one uses the specific property of the natural Quillen model on **Cat** that every object π is fibrant, which precisely means that the unique functor $\pi \to \mathbf{1}$ is an isofibration.

Exponentials (dereliction, weakening).

The promotion rules will be treated in dedicated section 5, but it is already easy to describe the rest of the rules. Consider one proof $\pi \vdash \Gamma, A$ interpreted as a separately fibrant span

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi \twoheadrightarrow A$$

the proof $\vdash \Gamma, ?A$ obtained after a dereliction rule (?) is interpreted by keeping the same underlying category and postcomposing the span

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi \twoheadrightarrow A \xrightarrow{\eta_A} \mathbf{Sym}\, A$$

with the isofibration $\eta_A : A \to \mathbf{Sym}\, A$ which sends any object a of A to the singleton word $\langle a \rangle$. Similarly, given a proof $\pi \vdash \Gamma$ interpreted as a separately fibrant span, the proof $\pi' \vdash \Gamma, ?A$ obtained after the weakening rule (weak) is interpreted by projecting π on the terminal category $\mathbf{1}$ with the isofibration $\pi \to \mathbf{1}$ and then composing

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi \twoheadrightarrow \mathbf{1} \twoheadrightarrow \mathbf{Sym}\, A$$

with the isofibration $\mathbf{1} \to \mathbf{Sym}\, A$ which sends the unique object $*$ of $\mathbf{1}$ to the empty word $\langle\rangle$ in $\mathbf{Sym}\, A$.

Exponentials (contraction).

For the contraction, we follow a treatment similar to that of the $\parr$.

Given a proof $\pi \vdash \Gamma\,,\, ?A\,,\, ?A$ and its interpretation as a separately fibrant span

$$\Gamma \overset{f}{\twoheadleftarrow\!\!\circ\!\!-} \pi \quad \begin{array}{l} \xrightarrow{f} \mathbf{Sym}_?A \\ \xrightarrow[g]{} \mathbf{Sym}_?A \end{array}$$

we need to define an isofibration $\Gamma \twoheadleftarrow\!\!\circ\!\!- \mathbf{contr}(f,g) \twoheadrightarrow \mathbf{Sym}_?A$. We first use the concatenation $\mathbf{Sym}_?A \times \mathbf{Sym}_?A \longrightarrow \mathbf{Sym}_?A$ to obtain:

$$\pi \xrightarrow{\langle f,g \rangle} \mathbf{Sym}_?A \times \mathbf{Sym}_?A \xrightarrow{\mathbf{concat}} \mathbf{Sym}_?A$$

which, however, is not an isofibration. Again, we consider its canonical fibrant replacement:

$$\pi \overset{\sim}{\rightarrowtail} \mathbf{contr}(f,g) \twoheadrightarrow \mathbf{Sym}_?A$$

which moreover comes with a fibration $\mathbf{contr}(f,g) \longrightarrow\!\!\!\!\rightarrow \pi$. The interpretation of the results of applying the contraction rule to π is then obtained by taking the category $\mathbf{contr}(f,g)$ as support and postcomposing

$$\Gamma \twoheadleftarrow\!\!\circ\!\!- \pi \xleftarrow{\sim} \mathbf{contr}(f,g) \longrightarrow\!\!\!\!\rightarrow \mathbf{Sym}_? A$$

with the context maps of the original separately fibrant span.

4 The shuffle factorization theorem

Before carrying on, we find convenient to introduce two different notations for the symmetric strict monoidal category $\mathbf{Sym}\,A$ freely generated by a small category A:

$\mathbf{Sym}_!\,A$ with objects finite words of objects of A noted $[a_1,\ldots,a_n]$;

$\mathbf{Sym}_?\,A$ with objects finite words of objects of A noted $\langle a_1,\ldots,a_n\rangle$.

Morphisms in $\mathbf{Sym}_?A$ are the same as before in $\mathbf{Sym}A$, but a morphism in $\mathbf{Sym}_!(A)$

$$[f_1,\ldots,f_n;\sigma] \quad : \quad [a_1,\ldots,a_n] \longrightarrow [a'_1,\ldots,a'_n]$$

is a tuple consisting of a permutation $\sigma \in \mathfrak{S}_n$ together with a family of morphisms $f_i : a_i \to a'_{\sigma(i)}$ in A. Our reason for having distinct conventions is that we see $\mathbf{Sym}_!(A)$ as a subcategory of the free cocartesian category $\mathbf{Fam}_+(A)$ generated by A and $\mathbf{Sym}_?(A)$ as a subcategory of the free cartesian category $\mathbf{Fam}_\times(A)$ generated by A. We then choose to redefine the interpretation of exponential formulae, setting $[\![?A]\!] = \mathbf{Sym}_?[\![A]\!]$ and $[\![!A]\!] = \mathbf{Sym}_![\![A]\!]$.

Recall that the category $\mathbf{Fam}_\times(A)$ defines a Grothendieck opfibration

$$q \quad : \quad \mathbf{Fam}_\times A \longrightarrow \mathbf{Set}$$

Given a function $f : J \to I$ and a family $\langle a_i\rangle_{i\in I}$ indexed by I, we write

$$\mathsf{push}_f : \langle a_{i\in I}\rangle_{i\in I} \to \langle a_{f(j)}\rangle_{j\in J}$$

for the canonical map. We are now ready to define the notions of *shuffle* and *deshuffle isomorphisms* in the symmetric monoidal categories $\mathbf{Sym}_!A$ and $\mathbf{Sym}_?A$. As a symmetric monoidal category, $\mathbf{Sym}_!A$ can be seen as a symmetric operad with the same objects, also noted $\mathbf{Sym}_!A$, whose n-ary morphisms $x_1,\ldots x_n \to x$ are defined as the morphisms $x_1 \otimes \cdots \otimes x_n \to x$ of the category. Given a sequence of n natural numbers $k_1,\ldots,k_n$ with sum the natural number $k = k_1 + \cdots + k_n$ seen as totally ordered sets, a $(k_1,\ldots,k_n)$-shuffle σ is a bijective function

$$k_1 + \cdots + k_n \xrightarrow{\sigma} k$$

such that each restriction

$$k_i \xrightarrow{\mathbf{inj}_i} k_1 + \cdots + k_n \xrightarrow{\sigma} k$$

is a monotone function. A *shuffle isomorphism* in the operad $\mathbf{Sym}_!A$ is defined as a morphism

$$[\mathbf{id},\ldots,\mathbf{id};\sigma] \quad : \quad [a^1_1,\ldots,a^1_{k_1}]\,,\,\cdots\,,\,[a^n_1,\ldots,a^n_{k_n}] \longrightarrow [a'_1,\ldots,a'_{k_1+\cdots+k_n}]$$

where the permutation $\sigma \in \mathfrak{S}_{k_1+\cdots+k_n}$ is a $(k_1,\ldots,k_n)$-shuffle.

Proposition 4.1 *Every n-ary morphism $f : x_1,\cdots,x_n \to y$ in the operad* $\mathbf{Sym}_!A$ *factors uniquely as $f = \sigma \circ (h_1,\cdots,h_n)$ where $\sigma : y_1,\cdots,y_n \to y$ is a shuffle isomorphism and each $h_k : x_k \to y_k$ is a unary morphism, for $1 \le k \le n$.*

Symmetrically, $\mathbf{Sym}_?A$ defines a symmetric cooperad where a *deshuffle isomorphism*

$$\langle\sigma;\mathbf{id},\ldots,\mathbf{id}\rangle \quad : \quad \langle a'_1,\ldots,a'_{k_1+\cdots+k_n}\rangle \longrightarrow \langle a^1_1,\ldots,a^1_{k_1}\rangle\,,\,\cdots\,,\,\langle a^n_1,\ldots,a^n_{k_n}\rangle$$

is defined as a n-ary morphism whose counterpart (or reversal) in the opposite symmetric operad $(\mathbf{Sym}_?A)^{op} = \mathbf{Sym}_!(A^{op})$ is a shuffle isomorphism.

Given a sequence of categories $\Gamma = A_1, \ldots, A_n$, we observe that the product category $\mathbf{Sym}_?\Gamma^\times = \mathbf{Sym}_?A_1 \times \cdots \times \mathbf{Sym}_?A_k$ is a symmetric strict monoidal category, with tensor product defined by point-wise list concatenation. The shuffle factorization system lift to $\mathbf{Sym}_?\Gamma^\times$ componentwise. A deshuffle isomorphism in $\mathbf{Sym}_?\Gamma^\times$ consists of a tuple $(\sigma_1, \ldots, \sigma_n)$ where σ_i is a deshuffle isomorphism in $\mathbf{Sym}_?A_i$.

5 The promotion rule

We now consider the *promotion* rule, whose interpretation exploits the shuffle factorization observed in §4. Given a context $\Gamma = A_1, \ldots, A_n$ we use the notations $?\Gamma = ?A_1, \ldots, ?A_n$ and $\mathbf{Sym}_?\Gamma = \mathbf{Sym}_?A_1, \ldots, \mathbf{Sym}_?A_n$. Given a proof $\pi \vdash ?\Gamma, B$, we consider its interpretation as a separately fibrant span

$$\mathbf{Sym}_?\Gamma \twoheadleftarrow\!\overset{\boldsymbol{f}}{\circ}\!\!- \pi \xrightarrow{\;g\;}\!\!\!\!\to B$$

and describe how to build the separately fibrant span interpreting the result of the promotion of π:

$$\mathbf{Sym}_?\Gamma \twoheadleftarrow\!\circ\!- \pi^! \twoheadrightarrow \mathbf{Sym}_!B \;.$$

First, we observe that the functorial action of $\mathbf{Sym}_!$ preserves isofibrations: given an isofibration $f : A \twoheadrightarrow B$ the functor $\mathbf{Sym}_!f : \mathbf{Sym}_!A \longrightarrow \mathbf{Sym}_!B$ defined by the map $[a_1, \ldots, a_k] \mapsto [f(a_1), \ldots, f(a_k)]$ is also an isofibration. Hence, we have an isofibration

$$\mathbf{Sym}_!g \quad : \quad \mathbf{Sym}_!\pi \twoheadrightarrow \mathbf{Sym}_!B$$

However, we also need to deal with the context span $\boldsymbol{f} : \pi \multimap\!\!\!\twoheadrightarrow \mathbf{Sym}_?\Gamma$.

As a first approach, we could try to exploit the universal property of $\mathbf{Sym}_!\pi$. Given a symmetric (unbiased) monoidal category X and a functor $f : A \longrightarrow X$, by the fact that $\mathbf{Sym}_!\pi$ is the free symmetric strict monoidal category on π, we obtain a functor $f^\dagger : \mathbf{Sym}_?A \longrightarrow X$, induced by the map $\langle a_1, \ldots, a_k\rangle \mapsto f(a_1) \otimes \cdots \otimes f(a_k)$, making the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathbf{Sym}_?A \\ {\scriptstyle f}\downarrow & \swarrow_{f^\dagger} & \\ X & & \end{array} .$$

Hence, we can apply this construction to our span $\boldsymbol{f} : \pi \multimap\!\!\!\twoheadrightarrow \mathbf{Sym}_?\Gamma$, componentwise, and obtain a span

$$\boldsymbol{f}^\dagger \quad : \quad \mathbf{Sym}_!\pi \multimap \mathbf{Sym}_?\Gamma$$

However, the resulting span has no reason to be separately fibrant. A simple example is given by applying the construction to the identity $\mathbf{id}_{\mathbf{Sym}_!A}$ since $\mathbf{id}^\dagger_{\mathbf{Sym}_!A} : \mathbf{Sym}_!\mathbf{Sym}_!A \longrightarrow \mathbf{Sym}_!A$ coincides with the monad multiplication $\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\rangle \mapsto \boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_k$, which is not an isofibration in general.

In order to fix this, we shall perform a replacement of the span $\boldsymbol{f}^\dagger$ by an appropriate construction, that we shall call the *shuffle category* on $\boldsymbol{f}$, denoted as $\mathbf{Shuffle}_{\boldsymbol{f}}$. As we will see, this construction comes equipped with an acyclic cofibration (an injective on objects equivalence of categories) $\mathbf{Sym}_!\pi \overset{\sim}{\rightarrowtail} \mathbf{Shuffle}_{\boldsymbol{f}}$ and a separately fibrant span $\boldsymbol{f}^\ddagger : \mathbf{Shuffle}_{\boldsymbol{f}} \multimap\!\!\!\twoheadrightarrow \mathbf{Sym}_?\Gamma$ making the following diagram commute

$$\begin{array}{ccc} \mathbf{Sym}_!\pi & \overset{\sim}{\rightarrowtail} & \mathbf{Shuffle}_{\boldsymbol{f}} \\ & {\scriptstyle \boldsymbol{f}^\dagger}\searrow \quad \swarrow{\scriptstyle \boldsymbol{f}^\ddagger} & \\ & \mathbf{Sym}_?\Gamma & \end{array} \tag{12}$$

as well as with an acyclic isofibration $\mathbf{Shuffle}_{\boldsymbol{f}} \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow \mathbf{Sym}_!\pi$ whose postcomposition with $\mathbf{Sym}_! g$: $\mathbf{Sym}_!\pi \twoheadrightarrow \mathbf{Sym}_! B$ will give the remaining component of the separately fibrant span $\pi^!$.

The shuffle category

We associate to every span $\boldsymbol{f} : \pi \longrightarrow\!\!\!\!\circ\!\!\!\!\longrightarrow \mathbf{Sym}_?\Gamma$, the *shuffle category* $\mathbf{Shuffle}_{\boldsymbol{f}}$ defined as follows:

- an object of $\mathbf{Shuffle}_{\boldsymbol{f}}$ is a pair consisting of an object

$$[p_1, \ldots, p_k] \in \mathbf{Sym}_!\pi$$

  together with a componentwise deshuffle isomorphism

$$\varphi = (\varphi_1, \ldots, \varphi_n) \quad : \quad \gamma \longrightarrow \boldsymbol{f}(p_1) \otimes \ldots \otimes \boldsymbol{f}(p_k)$$

  from an object

$$\gamma \in \mathbf{Sym}_?\Gamma^\times = \mathbf{Sym}_? A_1 \times \cdots \times \mathbf{Sym}_? A_n$$

  where each φ_i is a deshuffle isomorphism in the category $\mathbf{Sym}_? A_i$, for $1 \leq i \leq n$.
- a morphism from an object
  $[p_1, \ldots, p_k] \in \mathbf{Sym}_!\pi$ with componentwise deshuffle isomorphism $\varphi : \gamma \to \boldsymbol{f}(p_1) \otimes \ldots \otimes \boldsymbol{f}(p_k)$
  to an object
  $[p'_1, \ldots, p'_k] \in \mathbf{Sym}_!\pi$ with componentwise deshuffle isomorphism $\psi : \gamma' \to \boldsymbol{f}(p'_1) \otimes \ldots \otimes \boldsymbol{f}(p'_k)$
  is defined as a triple consisting of a permutation $\sigma \in \mathfrak{S}_k$ together with

$$\text{a family of morphisms } h_i : p_i \to p'_{\sigma i} \text{ in the category } \pi \text{ for } 1 \leq i \leq k,$$

  and a morphism $h : \gamma \to \gamma'$ in $\mathbf{Sym}_?\Gamma^\times$ making the diagram commute:

$$\begin{array}{ccccc} \gamma & & \xrightarrow{\quad\varphi\quad} & & \boldsymbol{f}(p_1) \otimes \cdots \otimes \boldsymbol{f}(p_k) \\ {\scriptstyle h}\downarrow & & & & \downarrow{\scriptstyle \boldsymbol{f}(h_1)\otimes\cdots\otimes\boldsymbol{f}(h_k)} \\ \gamma' & \xrightarrow{\psi} & \boldsymbol{f}(p'_1) \otimes \cdots \otimes \boldsymbol{f}(p'_k) & \xrightarrow{\mathsf{push}_\sigma} & \boldsymbol{f}(p'_{\sigma(1)}) \otimes \cdots \otimes \boldsymbol{f}(p'_{\sigma(k)}) \end{array} \tag{13}$$

Note that the data of the morphism h is redundant since it always exists and is uniquely determined in the category $\mathbf{Sym}_?\Gamma^\times$ by the sequence of morphisms $h_1, \ldots, h_n$ because φ and ψ and push_σ are isomorphisms.

The acyclic cofibration

We now define the equivalence of categories

$$\mathbf{Sym}_!\pi \underset{\sim}{\overset{\sim}{\rightleftarrows}} \mathbf{Shuffle}_{\boldsymbol{f}} \quad . \tag{14}$$

The cofibration is defined as the functor that maps every word $[p_1, \ldots, p_k]$ to the same word equipped with the identity deshuffle isomorphism

$$\mathbf{id}_\gamma \quad : \quad \boldsymbol{f}(p_1) \otimes \cdots \otimes \boldsymbol{f}(p_k) \longrightarrow \boldsymbol{f}(p_1) \otimes \cdots \otimes \boldsymbol{f}(p_k)$$

from the object $\gamma = \boldsymbol{f}(p_1) \otimes \cdots \otimes \boldsymbol{f}(p_k)$. The functor is clearly injective on objects, and fully faithful. It is also essentially surjective: each of a word and a deshuffle isomorphism is isomorphic to the same word equipped with the identity deshuffle isomorphism.

The fibration is defined as the forgetful functor which maps a pair $([p_1, \ldots, p_k], \varphi)$ to the underlying tuple $[p_1, \ldots, p_k]$. This functor is an isofibration. Indeed, given a pair $([p_1, \ldots, p_k], \varphi)$ consisting of a tuple $[p_1, \ldots, p_k]$ with deshuffle componentwise isomorphism $\varphi : \gamma \to \boldsymbol{f}(p_1) \otimes \ldots \otimes \boldsymbol{f}(p_k)$, together with an isomorphism $[h_1, \ldots, h_k; \sigma] : [p_1, \ldots, p_k] \to [p'_1, \ldots, p'_k]$ in $\mathbf{Sym}_!\pi$, with $h_i : p_i \to p'_{\sigma(i)}$ for $1 \leq i \leq k$, it is sufficient to consider the object $[p'_1, \ldots, p'_k]$ together with the identity deshuffle with the appropriate definition of h.

The fibrations (= isofibrations)

We now define the separately fibrant span mentioned above in (12)

$$\boldsymbol{f}^\ddagger \quad : \quad \mathbf{Shuffle}_{\boldsymbol{f}} \longrightarrow\!\!\!\!\!\!\circ\!\!\!\!\twoheadrightarrow \mathbf{Sym}_?\Gamma \tag{15}$$

by the map

$$([p_1, \ldots, p_k], \varphi : \gamma \to f(p_1) \otimes \cdots \otimes f(p_k)) \qquad \mapsto \qquad \gamma.$$

We now prove that this functor is an isofibration. We fix an object $([p_1, \ldots, p_k], \varphi)$ of $\mathbf{Shuffle}_{\boldsymbol{f}}$ consisting of a word $[p_1, \ldots, p_k]$, an object $\gamma = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \in \mathbf{Sym}_?\Gamma^\times$ and a componentwise shuffle isomorphism $\varphi : \gamma \to \boldsymbol{f}(p_1) \otimes \cdots \otimes \boldsymbol{f}(p_k)$. Suppose given a family of isomorphisms $\alpha_i : \boldsymbol{a}'_i \to \boldsymbol{a}_i$ in $\mathbf{Sym}_? A_i$ for $1 \leq i \leq n$. This induces an isomorphim

$$\psi = (\alpha_1, \ldots, \alpha_n) : \gamma' = (\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_n) \to \gamma = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$$

in the category $\mathbf{Sym}_?\Gamma^\times$. By the shuffle factorization system described in §4, the composite isomorphism $\varphi \circ \psi$ can be factored as

$$\varphi \circ \psi \;=\; \psi' \circ (\psi_1 \otimes \cdots \otimes \psi_k)$$

where

$$\psi' \quad : \quad \gamma' \longrightarrow \gamma_1 \otimes \cdots \otimes \gamma_k$$

is a componentwise deshuffle isomorphism and $\psi_i : \gamma_i \to \boldsymbol{f}(p_i)$ is a family of isomorphisms. At this stage, we use fact that $\boldsymbol{f}$ is separately fibrant, and deduce that there exists $p'_i \in \pi$ and isomorphims $h_i : p'_i \to p_i$ such that $\boldsymbol{f}(p'_i) = \gamma_i$ and $\boldsymbol{f}(h_i) = \psi_i$. We define an object of $\mathbf{Shuffle}_{\boldsymbol{f}}$ consisting of the word

$$[p'_1, \ldots, p'_k] \quad \in \quad \mathbf{Sym}_!\pi \tag{16}$$

equipped with the componentwise deshuffle isomorphism $\psi' \quad : \quad \gamma' \longrightarrow \boldsymbol{f}(p'_1) \otimes \cdots \otimes \boldsymbol{f}(p'_k)$. Then we define the isomorphims between $([p'_1, \ldots, p'_k], \psi')$ and $([p_1, \ldots, p_k], \varphi)$ as the word $[h_1, \ldots, h_k; id]$ together with the isomorphism ψ.

6 Conclusion

The homotopy interpretation of linear logic based on spans [27] associates to every proof of linear logic a *homotopy invariant* modulo cut-elimination, where homotopy is understood as categorical equivalence between categories in **Cat**. In the present paper, we give a detailed and direct description of the homotopy interpretation of proofs, where the *fibrant interpretation* used in [27] is replaced by a more liberal *separately fibrant interpretation* carefully chosen to remain as simple as possible. The construction relies on the existence of a number of simple and explicit separately fibrant replacements of the functors (concatenation tensor product, distributivity) underlying the rules of linear logic (in particular, promotion).

One purpose – and in fact the initial motivation – of this explicit reconstruction of the original model [27] is to investigate how the syntactic presentations [33,29] of the distributor interpretation of linear logic [13], could be adapted to the homotopy span model. A decisive step in the process of designing such a syntactic counterpart is to rely on a presentation as simple and explicit as possible of the homotopy span model. In that perspective, an important challenge for future work will be to understand how our combinatorial study of shuffles could be combined and unified with the analysis of symmetries developed by Clairambault and Forest, where an explicit construction of homotopy pullbacks using orthogonality is described in a variant of the span bicategory [5,6].

Also, our interpretation of linear logic based on separately fibrant spans has a definite polycategorical flavour. We plan to investigate more in detail the axiomatics of the 2-dimensional polycategorical structure of the interpretation. A complementary direction for future investigations would be to work at a tricategorical level and to define the 2-cells up to an isomorphism (building on [22]), instead of strictly as done in [27] and here. This alternative definition would require to compose spans using homotopy pullbacks instead of usual pullbacks, with tentative links to the recent work on the groupoid span interpretation of linear logic [5] and with the ∞-categorical construction of [20]. Another perspective would be to replace the **Sym** monad with another monoidal construction on small categories, such as the finite product completion. This variation has already been considered in the case of distributors [15,16,28] and it would thus be interesting to see how it works on the homotopy span model.

References

[1] Aguiar, M. and S. Mahajan, *Monoidal Functors, Species and Hopf Algebras*, volume 29 of *CRM Monograph Series*, American Mathematical Society (2010).

[2] Bucciarelli, A. and T. Ehrhard, *On phase semantics and denotational semantics in multiplicative-additive linear logic*, Ann. Pure Appl. Log. **102**, pages 247–282 (2000).
https://doi.org/10.1016/S0168-0072(99)00040-8

[3] Bucciarelli, A. and T. Ehrhard, *On phase semantics and denotational semantics: the exponentials*, Ann. Pure Appl. Log. **109**, pages 205–241 (2001).
https://doi.org/10.1016/S0168-0072(00)00056-7

[4] Burroni, A., *t-catégories (catégories dans un triple)*, Cahiers de topologie et géométrie différentielle **12**, pages 215–321 (1971).

[5] Clairambault, P. and S. Forest, *The cartesian closed bicategory of thin spans of groupoids*, in: *LICS*, pages 1–13 (2023).
https://doi.org/10.1109/LICS56636.2023.10175754

[6] Clairambault, P. and S. Forest, *An analysis of symmetry in quantitative semantics*, in: P. Sobocinski, U. D. Lago and J. Esparza, editors, *LICS*, ACM (2024).
https://doi.org/10.1145/3661814.3662092

[7] de Carvalho, D., *Execution time of λ-terms via denotational semantics and intersection types*, Math. Struct. Comput. Sci. **28**, pages 1169–1203 (2018).
https://doi.org/10.1017/S0960129516000396

[8] Ehrhard, T., *A completeness theorem for symmetric product phase spaces*, J. Symb. Log. **69**, pages 340–370 (2004).
https://doi.org/10.2178/JSL/1082418530

[9] Ehrhard, T., *Finiteness spaces*, Math. Struct. Comput. Sci. **15**, pages 615–646 (2005).
https://doi.org/10.1017/S0960129504004645

[10] Ehrhard, T., *An introduction to differential linear logic: proof-nets, models and antiderivatives*, Math. Struct. Comput. Sci. **28**, pages 995–1060 (2018).
https://doi.org/10.1017/S0960129516000372

[11] Ehrhard, T. and L. Regnier, *Differential interaction nets*, Theor. Comput. Sci. **364**, pages 166–195 (2006).
https://doi.org/10.1016/J.TCS.2006.08.003

[12] Fiore, M., Z. Galal and H. Paquet, *Stabilized profunctors and stable species of structures*, Log. Methods Comput. Sci. **20** (2024).
https://doi.org/10.46298/LMCS-20(1:17)2024

[13] Fiore, M., N. Gambino, M. Hyland and G. Winskel, *The cartesian closed bicategory of generalised species of structures*, Journal of the London Mathematical Society **77**, pages 203–220 (2007), ISSN 0024-6107.
https://doi.org/10.1112/jlms/jdm096

[14] Fiore, M., N. Gambino, M. Hyland and G. Winskel, *Relative pseudomonads, kleisli bicategories, and substitution monoidal structures*, Selecta Mathematica (2017).
https://doi.org/10.1007/s00029-017-0361-3

[15] Fiore, M. and A. Joyal, *Theory of para-toposes* (2015). Talk at the Category Theory Conference.

[16] Galal, Z., *A profunctorial scott semantics*, in: *5th International Conference on Formal Structures for Computation and Deduction, FSCD 2020, June 29-July 6, 2020, Paris, France (Virtual Conference)*, volume 167 of *LIPIcs*, pages 16:1–16:18 (2020).
https://doi.org/10.4230/LIPICS.FSCD.2020.16

[17] Gambino, N. and A. Joyal, *On operads, bimodules and analytic functors*, Memoirs of the American Mathematical Society **249** (2017), ISSN 1947-6221.
https://doi.org/10.1090/memo/1184

[18] Girard, J., *Linear logic*, Theor. Comput. Sci. **50**, pages 1–102 (1987).
https://doi.org/10.1016/0304-3975(87)90045-4

[19] Grellois, C. and P. Melliès, *Relational semantics of linear logic and higher-order model checking*, in: S. Kreutzer, editor, *24th EACSL Annual Conference on Computer Science Logic, CSL 2015, September 7-10, 2015, Berlin, Germany*, volume 41 of *LIPIcs*, pages 260–276, Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2015).
https://doi.org/10.4230/LIPICS.CSL.2015.260

[20] Harington, E. and S. Mimram, ∞*-categorical models of linear logic*, in: M. Fernández, editor, *10th International Conference on Formal Structures for Computation and Deduction, FSCD 2025, July 14-20, 2025, Birmingham, UK*, volume 337 of *LIPIcs*, pages 23:1–23:20, Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2025).
https://doi.org/10.4230/LIPICS.FSCD.2025.23

[21] Hirschhorn, P. S., *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*, American Mathematical Society (2003).

[22] Hoffnung, A. E., *Spans in 2-categories: A monoidal tricategory* (2013). 1112.0560.
https://arxiv.org/abs/1112.0560

[23] Hovey, M., *Model categories*, volume 63 of *Mathematical Surveys and Monographs*, American Mathematical Society (1999).

[24] Hyland, M., *Some reasons for generalising domain theory*, Mathematical Structures in Computer Science **20**, page 239–265 (2010).
https://doi.org/10.1017/S0960129509990375

[25] Joyal, A. and M. Tierney, *Strong stacks and classifying spaces*, in: A. Carboni, M. C. Pedicchio and G. Rosolini, editors, *Category Theory*, pages 213–236, Springer Berlin Heidelberg, Berlin, Heidelberg (1991), ISBN 978-3-540-46435-8.

[26] Kobayashi, N. and C. L. Ong, *Complexity of model checking recursion schemes for fragments of the modal mu-calculus*, Log. Methods Comput. Sci. **7** (2011).
https://doi.org/10.2168/LMCS-7(4:9)2011

[27] Melliès, P.-A., *Template games and differential linear logic*, in: *LICS* (2019).

[28] Olimpieri, F., *Intersection Types and Resource Calculi in the Denotational Semantics of Lambda-Calculus*, Ph.D. thesis, Aix-Marseille Université (2020).

[29] Olimpieri, F., *Intersection type distributors*, in: *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021* (2021).
https://doi.org/10.1109/LICS52264.2021.9470617

[30] Paquet, H. and P. Saville, *Strong pseudomonads and premonoidal bicategories* (2023). 2304.11014.

[31] Rezk, C., *A model category for categories* (1996). Unpublished note.

[32] Terui, K., *Semantic evaluation, intersection types and complexity of simply typed lambda calculus*, in: A. Tiwari, editor, *23rd International Conference on Rewriting Techniques and Applications (RTA'12) , RTA 2012, May 28 - June 2, 2012, Nagoya, Japan*, volume 15 of *LIPIcs*, pages 323–338, Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2012).
https://doi.org/10.4230/LIPICS.RTA.2012.323

[33] Tsukada, T., K. Asada and C. L. Ong, *Generalised species of rigid resource terms*, in: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*, pages 1–12, IEEE Computer Society (2017).
https://doi.org/10.1109/LICS.2017.8005093