Non-cartesian guarded recursion with daggers

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Abstract

Guarded recursion is a framework allowing for a formalisation of streams in classical (as opposed to concurrent, probabilistic, quantum) programming languages. The latter take their semantics in cartesian closed categories. However, some programming paradigms do not take their semantics in a cartesian setting; this is the case for concurrency, reversible and quantum programming for example. In this paper, we focus on reversible programming through its categorical model in dagger categories, which are categories that contain an involutive operator on morphisms. We show how to introduce the framework of guarded recursion into dagger categories with sufficient structure for data types, also called dagger rig categories. First, given an arbitrary category, we build another category shown to be suitable for guarded recursion in multiple ways, via enrichment and fixed point theorems. We then study the interaction between this construction and the structure of dagger rig categories, aiming for reversible programming. Finally, we show that our construction is suitable as a model of higher-order reversible programming languages, such as symmetric pattern matching, to which we add guarded recursion features.

Keywords: Guarded recursion, reversible programming, categorical semantics.

1 Introduction

Most programming language take a sound and adequate interpretation in *cartesian closed* categories. However, cartesian closed categories do not model all computing paradigms: reversible computation takes a sound and adequate model [33,10,11] in *inverse categories* [34,32]. The category of sets and partial injections is a concrete such category, and is not cartesian closed. Another field of interest where the categories are not cartesian closed is *quantum* computing: *e.g.* the category of *completely positive maps* [47], which is *compact* closed, but not cartesian, followed by many others [48,27,44,14,31,13,49]. The categories mentioned before are models of mixed quantum computing. The ones that integrate pure quantum computing are usually based on Hilbert spaces and bounded linear maps [46,29].

Among the categories mentioned above, inverse categories and categories based on Hilbert spaces are *dagger* categories. The dagger is an involutive operation on morphisms. This means that dagger categories possess a sort of partial inverse structure; for example, the partial inverse of a partial function $f\colon \{0,1\} \to \{0,1\}$ (see on the left below) is also a partial function $f^\dagger\colon \{0,1\} \to \{0,1\}$ (see on the right below).

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \text{undefined} & \text{otherwise} \end{cases} \qquad f^\dagger(x) = \begin{cases} 0 & \text{if } x = 1 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{1}$$

Moreover, this partial inverse allows us to provide an interpretation to functions even if the category is not closed [33,10,11]. A dagger category equipped with sufficient structure to interpret basic data types ($\otimes$ for pairs, $\oplus$ for control) is called a *dagger rig* category. Dagger rig categories are believed to be the canonical model of reversible computation [8].

While the notions of inductive types and recursion are well-studied in cartesian closed categories [1,23], this is not the case for dagger categories. Some families of dagger categories, such as inverse categories, are suitable to model inductive types and recursion, through their enrichment [32] in directed complete partial orders – a cartesian closed structure capable of solving recursive domain equations [24]. Inverse categories being in general not closed, the notion of enrichment is central: terms and morphisms typing judgements are interpreted in the enrichment and not in the externalisation, *i.e.* the underlying inverse category. The enrichment in dcpos provides a fixed point operator, therefore providing a model of reversible programming languages which are Turing complete [11]. However, the story does not go so well for Hilbert spaces, which lack the proper enrichment [28, Proposition 2.10]. Even if inductive types can be interpreted in Hilbert spaces [3, Theorem 3.2], it is an open question whether they can interpret recursion. We choose not to tackle this open question in this paper, and we instead suggest a different path, through guarded recursion.

Guarded recursion allows us to capture and control recursive calls within the type system [43], with the help of a modality, written $\blacktriangleright$ and called *later*. While this framework has been extensively studied for classical computation (as opposed to *e.g.* reversible or quantum computation) in cartesian closed settings, it is yet to be applied to less traditional ways of computation.

In the search for a denotational semantics of guarded recursion, a first account of solution to solve guarded domain equations was given within sheaves [19,20]. Then an adequate denotational semantics was given with ultrametric spaces [6], and later the same authors provided a more general semantics within the *topos of trees* [5]. The topos of trees is a category that is cartesian closed, which means in particular that it is a model of the simply-typed λ-calculus and of most traditional programming languages.

Several papers have then used synthetic guarded domain theory to develop refined models of guarded types or guarded recursion [15,41]. Guarded recursion has also been generalised to the case of causal structures [4] in a cartesian closed setting.

In the literature, guarded traced categories [25] are an instance of non-cartesian study of guarded recursion. The construction in this paper does not necessary yield a guarded trace category, offering a different angle in generalising guarded recursion.

The goal of this paper is to show that guarded recursion has a use beyond the formalisation of streams in classical programming languages, in particular for reversibility, through dagger categories. The study of the dagger structure is central in forming a link between the semantics of reversible programming languages and the semantics of guarded recursion in the topos of trees. Starting from an arbitrary category, we show that we can construct a guarded model out of this category, enriched in the topos of trees, therefore allowing for a model of guarded fixed points. Aiming at reversible programming, we study the interaction between the *dagger rig* structure and our guarded construction, in order to use the latter as interpretation for a reversible programming language with guarded recursion.

Content of the paper. The paper starts with a brief summary of classical guarded recursion. We picture the corresponding syntax (see §2.1) through the guarded λ-calculus [15], and then we introduce its interpretation in the topos of trees (see §2.2). We then present the *guarded construction* (see §3): starting from an arbitrary category with a terminal object, we construct a categorical model of guarded recursion. We show that the constructed category has characteristics similar to the topos of trees (see §3.1), and that it is even a model of guarded fixed points (see §3.2), allowing for the interpretation of guarded inductive types. In the following section, we focus on categorical models of reversible programming. We recall the definition of a dagger rig category (see §4.1). We show that the guarded construction preserves the rig structure (see §4.2), but we need to refine the construction to study its potential dagger structure (see §4.3). Finally, we show that the guarded construction from a dagger rig category is a categorical model of a reversible programming language with guarded recursion (see §5).

2 Classical Guarded Recursion

We provide an introduction to the concepts of guarded recursion, first through the syntax [15] (see §2.1), and then through its denotational semantics (see §2.2) in the *topos of trees* [5].

2.1 Syntactically

The initial goal of guarded recursion [43] is to ensure that functions defined on coinductive data types (*e.g.* streams) are well-defined. This is done with a modality, called *later*, and recursive calls can only be nested under the constructor associated to this modality.

We use the symbol $\blacktriangleright$ in the syntax for this modality. Semantically, this modality is represented by a functor (see §2.2). An excerpt of a type system is given below, where the statement 'X guarded in A' means that X is under a later modality $\blacktriangleright$ in the expression, *e.g.* in the type $1 \oplus \blacktriangleright X$, which gives the type of guarded natural numbers $\mu X.1 \oplus \blacktriangleright X$. Given a type A, we can form the type of lists as $\mu X.1 \oplus (A \otimes \blacktriangleright X)$, which we sometimes write $[A]$.

$$\frac{}{\Theta, X \vdash X} \qquad \frac{\Theta \vdash A}{\Theta \vdash \blacktriangleright A} \qquad \frac{\Theta, X \vdash A \quad X \text{ guarded in } A}{\Theta \vdash \mu X.A}$$

This modality must be introduced into the terms in the syntax. In a λ-calculus fashion, such as in the guarded λ-calculus [15], we write $\texttt{next}$ for the constructor introducing the later modality. Here is an excerpt of the typing rules.

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \texttt{next}\, M : \blacktriangleright A} \qquad \frac{\Gamma \vdash M : A_i}{\Gamma \vdash \texttt{inj}_i\, M : A_1 \oplus A_2} \qquad \frac{\Gamma \vdash M : A[\mu X.A/X]}{\Gamma \vdash \texttt{fold}\, M : \mu X.A}$$

Example 2.1 Let us fix a type A. Within the type of lists $[A] = \mu X.1 \oplus (A \otimes \blacktriangleright X)$, the empty list is obtained with $\texttt{fold}\ (\texttt{inj}_1\ *)$, which we will shorten as $[\,]$. The syntax for the list with one element M of type A is $\texttt{fold}\ (\texttt{inj}_2\ (M \otimes (\texttt{next}\, \texttt{inj}_1\ *)))$, shorten as $M :: \texttt{next}\, [\,]$.

One of the key points of guarded recursion is fixed points. The usual fixed point operator, such as for a typed λ-calculus [45,26,23], has the following type: $\texttt{fix}^\lambda : (A \to A) \to A$. The corresponding operational rule usually *unfolds* the fixed point: $\texttt{fix}^\lambda\ M \to M(\texttt{fix}^\lambda\ M)$.

In guarded recursion, the fixed point is more restrictive and recursive calls are *guarded*. To do so, the type of a guarded fixed point is $\texttt{fix} : (\blacktriangleright A \to A) \to A$, and given $\cdot \vdash M : \blacktriangleright A \to A$, that is to say, a function that takes a guarded term as an input, we have the following operational rule: $\texttt{fix}\ M \to M(\texttt{next}\, \texttt{fix}\ M)$. This means that after applying the fixed point rule, the next application must happen one step 'in the future'.

Studying the operational behaviour of the terms is an important aspect of programming language theory, but it is not the focus of this paper. In particular, there are many interesting notions in guarded type theory that do not apply in this paper, such as productivity. We restrict ourselves to a *structural* perspective on programming languages, by analysing their categorical denotational semantics.

The syntax for guarded recursion, introduced above, admits a sound and adequate denotational semantics [15] in the *topos of trees*, which is a cartesian closed category with sufficient structure for guarded recursion, as explained in the next section.

2.2 Categorically

In this section, we recall the main categorical structure underlying guarded recursion, referred to as *synthetic guarded domain theory* [5]. We write $\mathbf{S}$ for the category $\mathbf{Set}^{\mathbb{N}^{op}}$, referred to as the *topos of trees*. The definition of this category involves the natural numbers $\mathbb{N}$ with the usual order, which is pictured below. We also picture its opposite category, $\mathbb{N}^{op}$.

$$\mathbb{N} :: \ 0 \xrightarrow{\leq} 1 \xrightarrow{\leq} 2 \xrightarrow{\leq} 3 \xrightarrow{\leq} \cdots \qquad \mathbb{N}^{op} :: \ 0 \xleftarrow{\leq} 1 \xleftarrow{\leq} 2 \xleftarrow{\leq} 3 \xleftarrow{\leq} \cdots$$

The notation $\mathbf{Set}^{\mathbb{N}^{op}}$ means that the objects of the category are functors $\mathbb{N}^{op} \to \mathbf{Set}$. A functor $X : \mathbb{N}^{op} \to \mathbf{Set}$ assigns to every object n of $\mathbb{N}$ (*i.e.* a natural number) a set $X(n)$ and to every morphism of $\mathbb{N}$ (such as $n \leq n{+}1$) a function r_n^X between the sets $X(n{+}1)$ and $X(n)$, and therefore X can be pictured in the category $\mathbf{Set}$:

$$X(0) \xleftarrow{r_0^X} X(1) \xleftarrow{r_1^X} X(2) \xleftarrow{r_2^X} X(3) \longleftarrow \cdots$$

Finally, morphisms in the category $\mathbf{S}$ are natural transformations between the functors $\mathbb{N}^{op} \to \mathbf{Set}$. If X and Y are functors $\mathbb{N}^{op} \to \mathbf{Set}$, a natural transformation $f \colon X \to Y$ is pictured in $\mathbf{Set}$ with the following commutative diagram:

$$\begin{array}{ccccccccc} X(0) & \xleftarrow{r_0^X} & X(1) & \xleftarrow{r_1^X} & X(2) & \xleftarrow{r_2^X} & X(3) & \longleftarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ Y(0) & \xleftarrow[r_0^Y]{} & Y(1) & \xleftarrow[r_1^Y]{} & Y(2) & \xleftarrow[r_2^Y]{} & Y(3) & \longleftarrow & \cdots \end{array}$$

This category is a *topos*, which means that it contains a lot of structure. In particular, it is a cartesian category, and the cartesian product is obtained pointwise and directly inherited from the one in $\mathbf{Set}$. Moreover, like any topos [40, §I.6, Proposition 1], the category $\mathbf{S}$ is closed.

In particular, the exponential $[X \to Y](-) \colon \mathbb{N}^{op} \to \mathbf{Set}$ is given by $\mathbf{S}(\text{ʎ}(-) \times X, Y)$ where $\text{ʎ} \colon \mathbb{N}^{op} \to \mathbf{S}$ is the Yoneda embedding (the use of the Japanese hiragana *"yo"* was democratised by Loregian [39]). In our case, the Yoneda embedding can be described as follows: given a natural number n, the functor $\text{ʎ}(n) \times X$ is the same as X but truncated after the n^{th} component. It is pictured in $\mathbf{Set}$ as: $X(0) \xleftarrow{r_0^X} \cdots \xleftarrow{r_{n-1}^X} X(n) \xleftarrow{\text{¡}} \emptyset \longleftarrow \cdots$ with $\text{¡}_X \colon \emptyset \to X$ being the unique map from the empty set. The elements of $\mathbf{S}(\text{ʎ}(n) \times X, Y)$ are thus truncated natural transformations, such as:

$$\begin{array}{ccccccccc} X(0) & \xleftarrow{r_0^X} & \cdots & \xleftarrow{r_{n-1}^X} & X(n) & \xleftarrow{\text{¡}} & \emptyset & \longleftarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \text{¡}} & & \\ Y(0) & \xleftarrow[r_0^Y]{} & \cdots & \xleftarrow[r_{n-1}^Y]{} & Y(n) & \xleftarrow[r_n^Y]{} & Y(n{+}1) & \longleftarrow & \cdots \end{array} \tag{2}$$

and therefore can be described as a finite family of functions between sets $(f_0, \ldots, f_n)$ and the function between sets $r_n^{[X \to Y]} \colon \mathbf{S}(\text{ʎ}(n{+}1) \times X, Y) \to \mathbf{S}(\text{ʎ}(n) \times X, Y)$ gets a family of $n{+}1$ functions and drops the last one.

Because $\mathbf{S}$ is a cartesian closed category, it is a model of the simply-typed λ-calculus. In the following, we introduce the structure related to guarded recursion and show that it allows us to model a *guarded* fixed point operator.

Definition 2.2 [Later functor in $\mathbf{S}$ [5, §2.1]] The later functor $L \colon \mathbf{S} \to \mathbf{S}$ is such that given an object X in $\mathbf{S}$, we have $LX(0) = 1$ (the terminal object) and $LX(n{+}1) = X(n)$; and given a morphism $\alpha \colon X \to Y$ in $\mathbf{S}$, we have $L\alpha_0 = !_1$ (the terminal map), and $(L\alpha)_{n+1} = \alpha_n$.

Simply put, this functor shifts all components to the right. If X is an objects in $\mathbf{S}$, the object LX is pictured in $\mathbf{Set}$ as follows: $1 \xleftarrow{!} X(0) \xleftarrow{r_0^X} X(1) \xleftarrow{r_1^X} X(2) \longleftarrow \cdots$ When we use a category as a model for a programming language, the objects are models for types, whereas morphisms are models for terms and functions in the language. The *later* functor can be interpreted in the syntax as a modality. We then show that this modality can be introduced by terms in the syntax.

Definition 2.3 [Next in $\mathbf{S}$ [5, §2.1]] We call *next* the natural transformation $\nu \colon id \Rightarrow L$ in $\mathbf{S}$ defined as $\nu_{X,0} = !_1$ and $\nu_{X,n+1} = r_n^X$ for all objects X.

This category $\mathbf{S}$ comes with a specific form of fixed point for terms. In the case of the λ-calculus for example, a fixed point operator is a family of morphisms $\text{fix}_X^\lambda \colon [X \to X] \to X$, such that $\text{eval}_{X,X} \circ \langle \text{id}_{[X \to X]}, \text{fix}_X^\lambda \rangle = \text{fix}_X^\lambda$, where $\text{eval}_{X,Y} \colon [X \to Y] \times X \to Y$ is the morphism that evaluates its first argument in its second argument. In $\mathbf{S}$, we have a different kind of fixed point operator, because it involves the later modality. We refer to it as the *guarded* fixed point operator.

Lemma 2.4 (Guarded fixed points [5, §2.3]) *In the category* $\mathbf{S}$*, there exists a family of morphisms* $\mathrm{fix}_X \colon [LX \to X] \to X$ *such that* $\mathrm{eval}_{LX,X} \circ \langle \mathrm{id}_{[LX \to X]}, \nu_X \circ \mathrm{fix}_X \rangle = \mathrm{fix}_X$*.*

Fixed points can actually be obtained [5] for a more general family of morphisms, that we call *contractive* morphisms. A morphism is contractive if it factorises through a component of the *next* natural transformation. Contractivity is stable under composition (even with a non-contractive morphism), by tensor product and by currying.

Besides fixed points, recursive domain equations can also be solved in $\mathbf{S}$ [5] for a certain class of functors – roughly the ones that involve the later functor, which we call *contractive* functors. We do not provide full detail on this point, but we show later that the categories we introduce – namely, categories for non-cartesian guarded recursion – can solve domain equations for inductive types in a similar manner as in $\mathbf{S}$.

3 Non-Cartesian Guarded Construction

This section aims at providing a categorical tool sufficient to model guarded recursion in a non-cartesian setting (see Examples 4.5 and 4.6). We exhibit a construction similar to the category $\mathbf{S}$ (see §2.2) which allows for the interpretation of guarded recursion. We also show that guarded domain equations can be solved in this setting. We work with categories of the form $\mathbf{C}^{\mathbb{N}^{op}}$, where $\mathbb{N}$ is the category of natural numbers starting from 0, with morphisms defined by the usual order on natural numbers, as in §2.2. Given an object X of $\mathbf{C}^{\mathbb{N}^{op}}$, that is, a functor $\mathbb{N}^{op} \to \mathbf{C}$, its image on the morphism $n \leq n{+}1$ is written $r_n^X \colon X(n{+}1) \to X(n)$, and is a morphism in $\mathbf{C}$. This object X of $\mathbf{C}^{\mathbb{N}^{op}}$ can be represented with a diagram:

$$X(0) \xleftarrow{r_0^X} X(1) \xleftarrow{r_1^X} X(2) \xleftarrow{r_2^X} X(3) \longleftarrow \cdots \tag{3}$$

and a morphism $f \colon X \to Y$ can be pictured with the following diagram in $\mathbf{C}$:

$$\begin{array}{ccccccccc} X(0) & \xleftarrow{r_0^X} & X(1) & \xleftarrow{r_1^X} & X(2) & \xleftarrow{r_2^X} & X(3) & \longleftarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ Y(0) & \xleftarrow[r_0^Y]{} & Y(1) & \xleftarrow[r_1^Y]{} & Y(2) & \xleftarrow[r_2^Y]{} & Y(3) & \longleftarrow & \cdots \end{array}$$

The category $\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}$ looks like a category of presheaves, but is not a topos or even cartesian closed in general. We also show in the following that it is not a dagger category. However, it does not prevent us from using the dagger from the underlying category $\mathbf{C}$.

Let us fix a category $\mathbf{C}$ with terminal object T. For all objects A, there is a unique morphism $!_A \colon A \to T$. We write $\mathbf{C}^\infty$ for the category $\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}$, that we refer to as the *guarded construction*. This construction preserves terminal objects (and in fact, all limits and colimits).

Lemma 3.1 *The object of* $\mathbf{C}^\infty$ *pictured in* $\mathbf{C}$ *as:* $T \xleftarrow{\mathrm{id}_T} T \xleftarrow{\mathrm{id}_T} T \xleftarrow{\mathrm{id}_T} T \longleftarrow \cdots$ *is a terminal object in* $\mathbf{C}^\infty$*.*

We now show how the guarded construction is related to the topos of trees $\mathbf{S}$.

Categories in computer science are usually *locally small*, meaning that given two objects A and B, there is a *set* of morphisms $A \to B$. Enrichment is the study of the structure of those sets of morphisms, which could be vector spaces or topological spaces for example (more detail can be found in [35,36,42]). It turns out that homsets of $\mathbf{C}^\infty$ can be seen as objects in $\mathbf{S}$ – *i.e.* sequences of sets with morphisms between them. The results on $\mathbf{S}$ presented above (see §2.2) can then be applied at the level of morphisms in $\mathbf{C}^\infty$.

Lemma 3.2 *The category* $\mathbf{C}^\infty$ *yields an* $\mathbf{S}$*-enriched category with the following data:*

- *objects are objects in* $\mathbf{C}^{\mathbb{N}^{\mathrm{op}}}$*;*

- *hom-objects* $\mathbf{C}^\infty(X,Y)$ *of* $\mathbf{S}$*, defined as:* $\mathbf{C}^\infty(X,Y)(n) = \{(f_0,\dots,f_n) \mid f\colon X \to Y \text{ in } \mathbf{C}^{\mathbb{N}^{\mathrm{op}}}\}$*, a set of truncated natural transformations, and* $\mathbf{C}^\infty(X,Y)(n{+}1) \to \mathbf{C}^\infty(X,Y)(n)$ *is given as the function that forgets the last element;*
- *for all objects* X*, a morphism* $\mathrm{id}_X\colon 1 \to \mathbf{C}^\infty(X,X)$*, which outputs truncated identity natural transformations;*
- *for all objects* X,Y,Z*, a morphism:* $\mathrm{comp}_{X,Y,Z}\colon \mathbf{C}^\infty(Y,Z) \times \mathbf{C}^\infty(X,Y) \to \mathbf{C}^\infty(X,Z)$*, which composes the truncated natural transformations.*

As we do not need much enriched category theory, we choose to keep the conversation at the level of category – where $\mathbf{C}^\infty$ has objects X,Y and morphism $f\colon X \to Y$, but we keep in mind the notation above for $\mathbf{C}^\infty(X,Y)$ as an object in $\mathbf{S}$ and the corresponding morphisms in $\mathbf{S}$ for identity and composition.

In the categorical semantics of a programming language, we provide an interpretation of types as objects in the category and of terms as morphisms. In our construction, the morphisms live in $\mathbf{C}^\infty(X,Y)$, which is an object of $\mathbf{S}$, so the semantics of the terms is given in a mathematical setting sufficient to interpret guarded recursion. The enrichment described above therefore can equip any categorical model $\mathbf{C}$ with guarded recursion, and $\mathbf{C}^\infty$ is also a model, assuming that the guarded construction $\mathbf{C}^\infty$ preserves the structure of $\mathbf{C}$ relevant to the semantics. We give an example of the kind of structure that the guarded construction preserves with dagger rig categories later in the paper (see §4).

3.1 The Guarded Structure of the Guarded Construction

A feature of categories of the form $\mathbf{C}^{\mathbb{N}^{op}}$, when $\mathbf{C}$ has a terminal object, is the later functor. Operationally, this functor works as a delay modality (see §2.1). It can be used to keep track of the depth of a term and the number of recursive calls. It shifts the diagrammatic view in $\mathbf{C}$ of an object X (see Diagram 3) a step to the right, and adds a terminal object on the left.

Definition 3.3 [Later functor] The later functor $L^{\mathbf{C}}\colon \mathbf{C}^\infty \to \mathbf{C}^\infty$ is such that for all objects X in $\mathbf{C}^\infty$, we have $L^{\mathbf{C}}X(0) = T$ (the terminal object) and $L^{\mathbf{C}}X(n{+}1) = X(n)$; and for all morphisms $f\colon X \to Y$, we have $(L^{\mathbf{C}}f)_0 = !_T$ and $(L^{\mathbf{C}}f)_{n+1} = f_n$.

We use the same letter L for any later functor when it is not ambiguous. If ambiguity arises, we use the notation $L^{\mathbf{Set}}\colon \mathbf{S} \to \mathbf{S}$ (because $\mathbf{S} = \mathbf{Set}^\infty$) and $L^{\mathbf{C}}$.

Let us recall that an $\mathbf{S}$-enriched functor is a functor whose action on morphisms are morphisms in $\mathbf{S}$. We see in the next section that being $\mathbf{S}$-enriched is one of the necessary conditions for a functor to admit a fixed point.

Lemma 3.4 *The functor* $L^{\mathbf{C}}\colon \mathbf{C}^\infty \to \mathbf{C}^\infty$ *is* $\mathbf{S}$*-enriched.*

The action on objects of the later functors are linked due to the enrichment.

Lemma 3.5 *If* X *and* Y *are two objects in* $\mathbf{C}^\infty$*, then we have* $L^{\mathbf{Set}}\mathbf{C}^\infty(X,Y) \cong \mathbf{C}^\infty(L^{\mathbf{C}}X, L^{\mathbf{C}}Y)$.

Like in the category $\mathbf{S}$, the delay embodied by the functor L can be introduced by a natural transformation, called *next*. This natural transformation helps us introduce the delay in a programming language, as the denotational semantics of a delayed program.

Definition 3.6 [Next] The *next* natural transformation $\nu^{\mathbf{C}}\colon \mathrm{id} \Rightarrow L^{\mathbf{C}}$ is such that if X is an object in $\mathbf{C}^\infty$, we have $\nu^{\mathbf{C}}_{X,0} = !_{X(0)}$ and $\nu^{\mathbf{C}}_{X,n+1} = r^X_n$. It can be observed as a commutative diagram in $\mathbf{C}$, where it maps the sequence representing X to the sequence of $L^{\mathbf{C}}X$ as follows:

$$\begin{array}{ccccccccc}
X(0) & \xleftarrow{r_0^X} & X(1) & \xleftarrow{r_1^X} & X(2) & \xleftarrow{r_2^X} & X(3) & \longleftarrow & \cdots \\
\downarrow{\scriptstyle !} & & \downarrow{\scriptstyle r_0^X} & & \downarrow{\scriptstyle r_1^X} & & \downarrow{\scriptstyle r_2^X} & & \\
T & \xleftarrow[!]{} & X(0) & \xleftarrow[r_0^X]{} & X(1) & \xleftarrow[r_1^X]{} & X(2) & \longleftarrow & \cdots
\end{array}$$

Remark 3.7 Similarly to the later functor, the natural transformation ν can be defined in $\mathbf{S}$ (see Definition 2.3) as well as in $\mathbf{C}^\infty$. If any ambiguity arises, we use the notation $\nu^{\mathbf{Set}} \colon \mathrm{id}_{\mathbf{S}} \Rightarrow L^{\mathbf{Set}}$ and $\nu^{\mathbf{C}} \colon \mathrm{id}_{\mathbf{C}^\infty} \Rightarrow L^{\mathbf{C}}$.

Lemma 3.8 *If X and Y are two objects in $\mathbf{C}^\infty$, then we have $\nu^{\mathbf{Set}}_{\mathbf{C}^\infty(X,Y)} \cong L^{\mathbf{C}}_{X,Y}$.*

3.2 Fixed points of Locally Contractive Functors

We start by making precise what we mean by fixed point of a functor.

Definition 3.9 [Fixed Point] Given an endofunctor $T \colon \mathbf{C}^\infty \to \mathbf{C}^\infty$, a fixed point of T is a pair $(X, \alpha \colon TX \to X)$ such that α is an isomorphism.

A *locally contractive* $\mathbf{S}$-functor is one whose morphism mapping is a contractive morphism in $\mathbf{S}$ (see §2.2). We show later that a locally contractive functor has a unique fixed point, up to isomorphism. The functor $L^{\mathbf{C}} \colon \mathbf{C}^\infty \to \mathbf{C}^\infty$ is an example of locally contractive functor; and given two $\mathbf{S}$-enriched functors $F, G \colon \mathbf{C}^\infty \to \mathbf{C}^\infty$ such that either F or G is locally contractive, then FG is locally contractive [5, Lemma 7.3].

Given a morphism $\alpha \colon X \to Y$ in $\mathbf{C}^\infty$, one says that α is an n-isomorphism if the first n components of α (that is to say, $\alpha_0, \dots, \alpha_{n-1}$) are isomorphisms.

Lemma 3.10 *A locally contractive functor maps an n-isomorphism to an $n{+}1$-isomorphism.*

We can now prove the next theorem, similar to the one in the literature [5].

Theorem 3.11 *If an endofunctor is locally contractive, then it has a fixed point.*

We generalise the notion of locally contractive functors: an $\mathbf{S}$-functor $G \colon (\mathbf{C}^\infty)^k \to \mathbf{C}^\infty$ is *locally contractive* if it is locally contractive on all variables.

Theorem 3.12 (Parameterised Fixed Point) *A locally contractive functor admits a parameterised fixed point. In detail, if we have a locally contractive functor $G \colon (\mathbf{C}^\infty)^{k+1} \to \mathbf{C}^\infty$, there is a pair $(G^{\curlyvee}, \phi^G)$ such that:*

- *$G^{\curlyvee} \colon (\mathbf{C}^\infty)^k \to \mathbf{C}^\infty$ is a locally contractive functor,*
- *$\phi^G \colon G \circ \langle \mathrm{id}, G^{\curlyvee} \rangle \Rightarrow G^{\curlyvee}$ is a natural isomorphism,*
- *for every object $\overrightarrow{F}$ in $(\mathbf{C}^\infty)^k$, the pair $(G^{\curlyvee}\overrightarrow{F}, \phi^G)$ is the fixed point of $G(\overrightarrow{F}, -)$.*

Remark 3.13 A stronger theorem can be stated [5, Theorem 7.5], which allows for the interpretation of recursive dependent types. In this paper, we work within categories that are not necessarily monoidal closed; therefore we choose our type system to have a single polarity, and we restrict ourselves to (co)inductive types instead of general recursive types.

We have shown that inductive domain equations can be solved in $\mathbf{C}^\infty$, without any specific assumption on $\mathbf{C}$. In the next section, we focus on reversible programming through its model in dagger rig categories, where *rig* stands for the data structure of tensors and sum types, and the *dagger* ensures that operations are reversible. We later use Theorem 3.12 with a subcategory of $\mathbf{C}$ (see §5.1), in order to have a better interaction with the dagger structure.

4 Dagger and Rig Structure in Guarded Models

In this section, we recall the main definitions and present some examples of dagger rig categories (see §4.1). We then study the effect of the guarded construction on a dagger rig category with a terminal object (that turns out to be a zero object because of the dagger). The rig structure generalises well to the guarded constructed category (see §3), but not the dagger (see Remark 4.10). In order to seize the interaction with the dagger structure, we refine our construction and show that our construction contains a sufficiently big subcategory that is also dagger rig (see §4.3). This category is then used in the rest of the paper, in particular to interpret reversible functions (see §5.3).

We study the interaction between the construction above and dagger rig categories.

4.1 Background: Dagger Rig Categories

We start with the notion of *dagger*. A category $\mathbf{C}$ is a *dagger category* if there is a contravariant endofunctor $(-)^\dagger \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ such that $X^\dagger = X$ for all objects X and $f^{\dagger\dagger} = f$ for all morphisms f in $\mathbf{C}$. A functor F between dagger categories is a *dagger functor* if $F(f^\dagger) = F(f)^\dagger$ for all morphisms f. As a model of computation, the dagger means that there is a sound way to *reverse* programs. Note that 'reversibility' in the context of reversible programming does not mean that all programs are bijections. They admit an inverse, but it may be partial.

Example 4.1 The category of sets and bijective functions is a dagger category. If the function $f \colon X \to Y$ is bijective, its dagger is $f^{-1} \colon Y \to X$.

Definition 4.2 Let $\mathbf{C}$ be a dagger category. A morphism $f \colon X \to Y$ is called:

- a *dagger monomorphism* if $f^\dagger \circ f = \mathrm{id}_X$;
- a *dagger epimorphism* if $f \circ f^\dagger = \mathrm{id}_Y$;
- a *dagger isomorphism* if it is both a dagger monomorphism and a dagger epimorphism.

Remark 4.3 A morphism that is dagger monic (resp. epic) is in particular split monic (resp. epic), and therefore is monic (resp. epic). However, the converse is not true in general.

To interpret computation, a category needs more structure. We assume that we need to form pairs of programs, represented by a monoidal structure $\otimes$, and conditions on programs, represented by a monoidal structure $\oplus$, *e.g.* the type of booleans given by $\mathtt{I} \oplus \mathtt{I}$.

A *dagger rig category* is a dagger category equipped with symmetric monoidal structures $(\otimes, I)$ and $(\oplus, O)$, such that $\otimes$ and $\oplus$ are dagger functors, with their coherence isomorphisms, and with additional natural dagger isomorphisms (satisfying coherence conditions [37]):

$$(X \oplus Y) \otimes Z \xrightarrow{\sim} (X \otimes Z) \oplus (Y \otimes Z), \quad O \otimes X \xrightarrow{\sim} O,$$

$$Z \otimes (X \oplus Y) \xrightarrow{\sim} (Z \otimes X) \oplus (Z \otimes Y), \quad X \otimes O \xrightarrow{\sim} O.$$

Example 4.4 The category of sets and relations is a dagger rig category. If $f \colon X \to Y$ is a relation between two sets, then $f^\dagger = \{(y, x) \mid (x, y) \in f\}$. This category is known to be a model of linear logic [21, Section 2.1] and of programming languages based on linear logic.

Example 4.5 We write **PInj** for the category of sets and partial injections. It is an inverse category [34], which means that it is equipped with a dagger and some equational conditions on the dagger. An example of a partial injection and its partial inverse is given in the introduction (1). Inverse categories have been generalised to *restriction categories* [16,17,18]. This family of categories is used to represent reversible computation, *e.g.* as a model [33] of Rfun [50], or as a model [10,11] of reversible pattern matching [46].

Example 4.6 Another example is the category of Hilbert spaces and linear maps, that we denote **Hilb**. Its wide subcategory whose morphisms are unitaries (or dagger isomorphisms) is used to interpret pure quantum operations [46,29,7,9,38], and its wide subcategory whose morphisms are isometries (or dagger monomorphisms) is a model of quantum states [38, §3.5]. Both are also rig dagger categories.

The categories **PInj** and **Hilb** are $\oplus$-semicartesian. This means that the unit of the monoidal structure associated to $\oplus$ is a terminal object – and therefore a zero object, since we are working within dagger categories. This fits the story of the models for guarded recursion where a terminal object is necessary to define the *later* functor (see Definition 3.3) and the natural transformation *next* (see Definition 3.6).

4.2 The Guarded Construction from a Dagger Rig Category

Fix $\mathbf{C}$ a $\oplus$-semicartesian dagger rig category, *i.e.* the monoidal unit O of $\oplus$ is a zero object. Thus there is a unique $!_X \colon X \to O$ and a unique $¡_X \colon O \to X$ for all objects X; therefore, we have $!_X \circ ¡_X = \mathrm{id}_O$

and $(¡_X)^\dagger = !_X$. The morphism $!_X \colon X \to O$ is thus a dagger epimorphism. We obtain the left injection $\iota_1 \colon X \to X \oplus Y$ by: $X \xrightarrow{(\rho_X^\oplus)^{-1}} X \oplus O \xrightarrow{\mathrm{id}_X \oplus ¡_Y} X \oplus Y$, and the left injection is therefore a dagger epimorphism as well.

Lemma 4.7 *The category $\mathbf{C}^\infty$ is a rig category. It is in particular $\oplus$-semicartesian.*

We detail the interaction between the guarded structure and the rig structure from $\mathbf{C}$.

Lemma 4.8 *If X and Y are objects in $\mathbf{C}^\infty$, then we have, with $\star \in \{\otimes, \oplus\}$:*

$$L^{\mathbf{C}}(X \star Y) \cong L^{\mathbf{C}} X \star L^{\mathbf{C}} Y, \quad \nu^{\mathbf{C}}_{X \star Y} \cong \nu^{\mathbf{C}}_X \star \nu^{\mathbf{C}}_X.$$

Corollary 4.9 *Let X_1 and X_2 be objects in $\mathbf{C}^\infty$. We have that $\nu^{\mathbf{C}}_{X_1 \oplus X_2} \circ \iota_i \cong \iota_i \circ \nu^{\mathbf{C}}_{X_i}$.*

However, the category $\mathbf{C}^\infty$ does not handle the dagger well, as highlighted below.

Remark 4.10 [Dagger in Guarded Construction] Morphisms in $\mathbf{C}^\infty$ are natural transformations whose components are morphisms in $\mathbf{C}$. In that regard, the category $\mathbf{C}^\infty$ inherits some of the structure of $\mathbf{C}$, but not all, *e.g.* the dagger. We however stick to the notation $f^\dagger$ for the *componentwise* dagger of $f \colon X \to Y$ in $\mathbf{C}^\infty$, even if it might not be a morphism in $\mathbf{C}^\infty$.

In order to characterise which morphisms admit a dagger, we choose to work in a less general category, defined as follows. First, we write $\mathbf{C}_e$ for the wide subcategory of $\mathbf{C}$ whose morphisms are only dagger epimorphisms. We then define the category $\mathbf{C}_{\mathrm{e}}^\infty$, whose objects are the objects of $(\mathbf{C}_e)^{\mathbb{N}^{\mathrm{op}}}$ and whose morphisms are natural transformations in $\mathbf{C}$. In other words, the category $\mathbf{C}_{\mathrm{e}}^\infty$ has cochains of dagger epimorphisms as objects and arbitrary natural transformations as morphisms. This means that the objects of $\mathbf{C}_{\mathrm{e}}^\infty$ are cochains of *larger and larger* objects. Note that $\mathbf{C}_{\mathrm{e}}^\infty$ is fully embedded in $\mathbf{C}^\infty$.

Similarly to $\mathbf{C}^\infty$, the category $\mathbf{C}_{\mathrm{e}}^\infty$ yields an $\mathbf{S}$-enriched category. However, this presentation of the enrichment does not account for the dagger. We choose to capture a more precise enrichment for $\mathbf{C}_{\mathrm{e}}^\infty$.

Lemma 4.11 *Due to the new restriction to dagger epimorphisms in the cochains, we can equivalently provide a description of the enrichment of $\mathbf{C}_{\mathrm{e}}^\infty$ in $\mathbf{S}$ with:*

$$\mathbf{C}_{\mathrm{e}}^\infty(X,Y)(n) = \{f_n \mid f \colon X \to Y \text{ in } \mathbf{C}^{\mathbb{N}^{\mathrm{op}}}\}, \quad \text{and} \quad \begin{cases} \mathbf{C}_{\mathrm{e}}^\infty(X,Y)(n+1) \to \mathbf{C}_{\mathrm{e}}^\infty(X,Y)(n) \\ f \mapsto r_n^Y \circ f \circ (r_n^X)^\dagger \end{cases}$$

Remark 4.12 The embedding $E \colon \mathbf{C}_{\mathrm{e}}^\infty \hookrightarrow \mathbf{C}^\infty$ is $\mathbf{S}$-enriched.

We later use this category as a model for a guarded reversible programming language (see §5). To do so, the category $\mathbf{C}_{\mathrm{e}}^\infty$ needs to preserve structure both linked to guarded recursion and the monoidal tensors. The interaction with the dagger is detailed in the next section.

Note that the later functor $L^{\mathbf{C}}$ (co)restricts to $\mathbf{C}_{\mathrm{e}}^\infty$ (if X is a cochain of dagger epimorphisms, $L^{\mathbf{C}} X$ is too). We can then keep the notation $L^{\mathbf{C}} \colon \mathbf{C}_{\mathrm{e}}^\infty \to \mathbf{C}_{\mathrm{e}}^\infty$ non ambiguously.

Lemma 4.13 *The functor $L^{\mathbf{C}} \colon \mathbf{C}_{\mathrm{e}}^\infty \to \mathbf{C}_{\mathrm{e}}^\infty$ is $\mathbf{S}$-enriched.*

In the same way as $\mathbf{C}^\infty$ (see Lemma 4.7), the category $\mathbf{C}_{\mathrm{e}}^\infty$ inherits monoidal structures from the underlying category $\mathbf{C}$, and the resulting monoidal structures in $\mathbf{C}_{\mathrm{e}}^\infty$ are pointwise.

Lemma 4.14 *The category $\mathbf{C}_{\mathrm{e}}^\infty$ is a rig category. It is in particular $\oplus$-cartesian.*

We then show that $\mathbf{C}_{\mathrm{e}}^\infty$ has an interaction with the dagger in $\mathbf{C}$, and thus contains more structure inherited from $\mathbf{C}$. We can then point out morphisms in $\mathbf{C}_{\mathrm{e}}^\infty$ that can be reversed.

4.3 Guarded Dagger Category

As explained in Remark 4.10, thus the dagger notation $(-)^\dagger$ in the guarded construction is loose. However, this notation can be used in some cases (see Lemma 4.15). We then point out which morphisms in $\mathbf{C}_\mathrm{e}^\infty$ are such that their pointwise dagger is also a morphism in $\mathbf{C}_\mathrm{e}^\infty$.

Lemma 4.15 *If $f\colon X \to Y$ is in $\mathbf{C}_\mathrm{e}^\infty$, then $\nu_Y^{\mathbf{C}} \circ f \circ \left(\nu_X^{\mathbf{C}}\right)^\dagger : LX \to LY$ is in $\mathbf{C}_\mathrm{e}^\infty$.*

By Lemma 3.5, *later* at the level of a hom-object $\mathbf{C}^\infty(X,Y)$ is equivalent to *later* applied to both objects X and Y. A similar statement holds in $\mathbf{C}_\mathrm{e}^\infty$ for *next* due to the dagger.

Lemma 4.16 *Given X and Y two objects of $\mathbf{C}_\mathrm{e}^\infty$, we have $\nu^{\mathbf{Set}}_{\mathbf{C}_\mathrm{e}^\infty(X,Y)} = \nu_Y^{\mathbf{C}} \circ - \circ \left(\nu_X^{\mathbf{C}}\right)^\dagger$.*

This also means that if $f\colon X \to Y$ is a morphism in $\mathbf{C}_\mathrm{e}^\infty$, then $\nu^{\mathbf{Set}}_{\mathbf{C}_\mathrm{e}^\infty(X,Y),n+1}(f_{n+1}) = f_n$.

The category $\mathbf{C}_\mathrm{e}^\infty$ is not a dagger category, but if it were, the dagger of a morphism in $\mathbf{C}_\mathrm{e}^\infty$ would necessarily be the dagger taken pointwise on the components in $\mathbf{C}$ – the same way the inverse of an isomorphism is pointwise.

Definition 4.17 [Daggerable] If $f\colon X \to Y$ is a morphism in $\mathbf{C}_\mathrm{e}^\infty$, we say that it *admits a dagger* or is *daggerable* if the family $\{f_n^\dagger\}_{n\in\mathbb{N}}$ of morphisms in $\mathbf{C}$ verifies:

$$\begin{array}{ccccccccc}
Y(0) & \xleftarrow{r_0^Y} & Y(1) & \xleftarrow{r_1^Y} & Y(2) & \xleftarrow{r_2^Y} & Y(3) & \longleftarrow & \cdots \\
\downarrow{\scriptstyle f_0^\dagger} & & \downarrow{\scriptstyle f_1^\dagger} & & \downarrow{\scriptstyle f_2^\dagger} & & \downarrow{\scriptstyle f_3^\dagger} & & \\
X(0) & \xleftarrow[r_0^X]{} & X(1) & \xleftarrow[r_1^X]{} & X(2) & \xleftarrow[r_2^X]{} & X(3) & \longleftarrow & \cdots
\end{array}$$

and therefore yields a morphism in $\mathbf{C}_\mathrm{e}^\infty$, that we write $f^\dagger\colon Y \to X$. We write $\mathbf{C}_\dagger^\infty$ for the wide subcategory of $\mathbf{C}_\mathrm{e}^\infty$ whose morphisms are daggerable. It is by definition a rig category.

Example 4.18 Morphisms $\nu_X^{\mathbf{C}}$ are, in general, not daggerable.

While it is a difficult task to exactly characterise the morphisms of $\mathbf{C}_\dagger^\infty$, we can point out a sufficient number of them to interpret a reversible programming language.

Lemma 4.19 *Morphisms in $\mathbf{C}_\mathrm{e}^\infty$ with dagger isomorphisms components are daggerable.*

This implies, in particular, that identity morphisms are in $\mathbf{C}_\dagger^\infty$ – although this is already implied above when we refer to $\mathbf{C}_\dagger^\infty$ as a subcategory of $\mathbf{C}_\mathrm{e}^\infty$. More interestingly, coherence morphisms associated to the rig structure are defined pointwise, and therefore all their components are dagger isomorphisms. Thus, Lemma 4.19 is sufficient to ensure that the category $\mathbf{C}_\dagger^\infty$ is a rig category, with the structure inherited from the underlying category $\mathbf{C}$.

Lemma 4.20 *If $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ are morphisms in $\mathbf{C}_\dagger^\infty$, then:*

- *$f \otimes g\colon X_1 \otimes X_2 \to Y_1 \otimes Y_2$ and*
- *$f \oplus g\colon X_1 \oplus X_2 \to Y_1 \oplus Y_2$ are morphisms in $\mathbf{C}_\dagger^\infty$.*

The lemmas above show that $\mathbf{C}_\dagger^\infty$ is a dagger rig category. It is therefore a suitable model to interpret a simply-typed reversible programming language, and the following lemmas continue to push towards the semantics of a reversible language as expressive as possible.

Lemma 4.21 *The morphisms $!_X\colon X \to O$ and $¡_X\colon O \to X$ in $\mathbf{C}_\mathrm{e}^\infty$ admit a dagger.*

This ensures that injections $\iota_i\colon X_i \to X_1 \oplus X_2$, with $i \in \{1,2\}$, are daggerable, because they are formed as composition of daggerable morphisms (see §4.2).

Following Lemma 4.15, we can point out another important family of daggerable morphisms.

Lemma 4.22 *If $f : X \to Y$ in $\mathbf{C}_{\mathrm{e}}^{\infty}$ is daggerable, then $\nu_Y^{\mathbf{C}} \circ f \circ \left(\nu_X^{\mathbf{C}}\right)^{\dagger} : LX \to LY$ is also.*

The category $\mathbf{C}_{\dagger}^{\infty}$ is dagger rig and $\oplus$-semicartesian due to the observations above. This category is also enriched in $\mathbf{S}$, which means that the guarded fixed point operator can be used at the level of morphisms, with morphisms that are reversible, since they admit a dagger.

We have outlined a relevant structure for guarded recursion with daggers, and shown its link with the topos of trees $\mathbf{S}$ through an enrichment. We show in the following section that this enrichment of $\mathbf{C}_{\mathrm{e}}^{\infty}$ in $\mathbf{S}$ also provides fixed points for a certain class of functors. These fixed points of functors are useful in the denotational semantics of infinite data types.

5 Application: Semantics of Guarded Symmetric Pattern matching

Symmetric pattern matching [46] is a typed reversible language, based on Theseus [30], first introduced as a general framework for pure quantum programming – *i.e.* quantum computing with only reversible operations – which works with infinite data types, such as lists (which is unique in the pure quantum literature). The syntax of symmetric pattern matching is simple and close to the type system, and we see it as the λ-calculus for reversible programming.

There are several refinements of the original paper [46], such as connections with linear logic and infinitary proofs [12], a full denotational semantics and adequacy for its higher-order classical reversible fragment [10,11], and a full denotational semantics and completeness result for its first-order quantum fragment [38, Chapter 3]. The denotational semantics of the full higher-order quantum language is an open question (see [38, Section 5.2]).

The required structure to be an interpretation of higher-order pattern matching is:

(i) a rig structure, interpretating tensors and sum types;

(ii) parameterised fixed points for functors, allowing for guarded inductive types;

(iii) a *join* structure, to interpret iso abstractions, the first-order functions of the language;

(iv) a dagger structure, to account for the reversibility of first-order functions of the language;

(v) an enrichment in a cartesian closed category, permitting a higher-order language with a (guarded) λ-calculus on top of functions;

(vi) a fixed point operator in the enrichment category, to interpret (guarded) recursion.

We detail all those points below, with the corresponding syntax adapted to guarded recursion. The full syntax and typing rules are detailed in Figures 1 and 2. We fix a $\oplus$-semicartesian dagger rig category $\mathbf{C}$; we write $\mathbf{C}_{\mathrm{e}}^{\infty}$ for the category of sequences of dagger epimorphisms as objects and arbitrary natural transformations, and we write $\mathbf{C}_{\dagger}^{\infty}$ for its full subcategory of morphisms that admit a dagger.

5.1 Semantics of Ground Types

For the interpretation of types, we use the category $(\mathbf{C}_{\mathrm{e}})^{\infty}$, which is the guarded construction arising from $\mathbf{C}_e$, the subcategory of $\mathbf{C}$ whose morphisms are all dagger epimorphisms. Note that $(\mathbf{C}_{\mathrm{e}})^{\infty}$ is a wide subcategory of $\mathbf{C}_{\mathrm{e}}^{\infty}$, in the sense that they have the same objects.

We tackle Point (i) in Lemma 4.7 above. This allows for the introduction of sum types $\oplus$ and tensor products $\otimes$. Moreover, Theorem 3.12 ensures that locally contractive functors admit parameterised fixed points, fitting Point (ii). Therefore, inductive types $\mu X.A$ can be formed, under the condition that X is guarded in A (see §2.1). Together with the later functor, we have a suitable model for the type introduction rules below.

$$\frac{}{\Theta, X \vdash X} \quad \frac{}{\Theta \vdash \mathtt{I}} \quad \frac{\Theta \vdash A}{\Theta \vdash \blacktriangleright A} \quad \frac{\Theta \vdash A \quad \Theta \vdash B}{\Theta \vdash A \star B}\ {\scriptstyle \star \in \{\oplus, \otimes\}} \quad \frac{\Theta, X \vdash A \quad X \text{ guarded in } A}{\Theta \vdash \mu X.A}$$

The interpretation of a type judgement $\Theta \vdash A$ is a functor $[\![\Theta \vdash A]\!] : ((\mathbf{C}_{\mathrm{e}})^{\infty})^{|\Theta|} \to (\mathbf{C}_{\mathrm{e}})^{\infty}$. The denotation of $[\![\Theta \vdash \blacktriangleright A]\!]$ is obtained by postcomposing $[\![\Theta \vdash A]\!]$ with the later functor $L^{\mathbf{C}_e} : (\mathbf{C}_{\mathrm{e}})^{\infty} \to (\mathbf{C}_{\mathrm{e}})^{\infty}$, and the one of $[\![\Theta \vdash \mu X.A]\!]$ is $[\![\Theta, X \vdash A]\!]^{\curlyvee}$ (see Theorem 3.12).

$$
\begin{array}{lll}
\text{(Ground types)} & A, B & ::= \mathtt{I} \mid A \oplus B \mid A \otimes B \mid \blacktriangleright A \mid X \mid \mu X.A \\
\text{(Function types)} & T_1, T_2 & ::= A \leftrightarrow B \mid \blacktriangleright T \mid T_1 \to T_2 \\
\text{(Pairing)} & t, t_1, t_2 & ::= * \mid t_1 \otimes t_2 \\
\text{(Injections)} & & \mid \mathtt{inj}_1\ t \mid \mathtt{inj}_2\ t \\
\text{(Function application)} & & \mid \omega\ t \\
\text{(Later modality)} & & \mid \mathtt{next}\ t \\
\text{(Inductive terms)} & & \mid \mathtt{fold}\ t \\
\text{(Abstraction)} & \omega & ::= \{t_1 \mapsto t_1' \mid \cdots \mid t_m \mapsto t_m'\} \\
\text{(Higher modality)} & & \mid \mathtt{next}\ \omega \\
\text{(Fixed points)} & & \mid \phi \mid \mathrm{fix}\phi.\omega \\
\text{(Higher functions)} & & \mid \lambda\phi.\omega \mid \omega_2\omega_1
\end{array}
$$

Fig. 1. Syntax of guarded symmetric pattern matching.

$$
\frac{}{\Theta, X \vdash X} \quad \frac{}{\Theta \vdash \mathtt{I}} \quad \frac{\Theta \vdash A}{\Theta \vdash \blacktriangleright A} \quad \frac{\Theta \vdash A \quad \Theta \vdash B}{\Theta \vdash A \star B}\ {\scriptstyle \star \in \{\oplus, \otimes\}} \quad \frac{\Theta, X \vdash A \quad X \text{ guarded in } A}{\Theta \vdash \mu X.A}
$$

$$
\frac{\vdash A \quad \vdash B}{\vdash_\omega A \leftrightarrow B} \qquad \frac{\vdash_\omega T}{\vdash_\omega \blacktriangleright T} \qquad \frac{\vdash_\omega T_1 \quad \vdash_\omega T_2}{\vdash_\omega T_1 \to T_2}
$$

$$
\frac{}{\Psi; \cdot \vdash *\colon I} \qquad \frac{\Psi; \Delta_1 \vdash t_1 \colon A_1 \quad \Psi; \Delta_2 \vdash t_2 \colon A_2}{\Psi; \Delta_1, \Delta_2 \vdash t_1 \otimes t_2 \colon A_1 \otimes A_2}
$$

$$
\frac{\Psi; \Delta_1 \vdash t_1 \colon A \otimes B \quad \Psi; \Delta_2, x \colon A, y \colon B \vdash t_2 \colon C}{\Psi; \Delta_2, \Delta_1 \vdash \mathtt{let}\, x \otimes y = t_1\ \mathtt{in}\ t_2 \colon C} \qquad \frac{\Psi; \Delta \vdash t \colon A}{\Psi; \Delta \vdash \mathtt{next}\, t \colon \blacktriangleright A}
$$

$$
\frac{\Psi; \Delta \vdash t \colon A_i}{\Psi; \Delta \vdash \mathtt{inj}_i\ t \colon A_1 \oplus A_2} \qquad \frac{\Psi; \Delta \vdash t \colon A[\mu X.A/X]}{\Psi; \Delta \vdash \mathtt{fold}\, t \colon \mu X.A}
$$

$$
\frac{\Psi; \vdash_\omega \omega \colon \blacktriangleright(A \leftrightarrow B) \quad \Psi; \Delta \vdash t \colon \blacktriangleright A}{\Psi; \Delta \vdash \omega \circledcirc t \colon \blacktriangleright B} \qquad \frac{\Psi; \vdash_\omega \omega \colon A \leftrightarrow B \quad \Psi; \Delta \vdash t \colon A}{\Psi; \Delta \vdash \omega\ t \colon B}
$$

$$
\frac{\Psi; \Delta_i \vdash t_i \colon A \quad \forall i \neq j, t_i \perp t_j \quad \Psi; \Delta_i \vdash t_i' \colon B \quad \forall i \neq j, t_i' \perp t_j'}{\Psi; \vdash_\omega \{t_1 \leftrightarrow t_1' \ \mid\ \cdots \ \mid\ t_n \leftrightarrow t_n'\} \colon A \leftrightarrow B}
$$

$$
\frac{}{\Psi; \phi \colon T \vdash_\omega \phi \colon T} \qquad \frac{\Psi \vdash_\omega \omega \colon T}{\Psi \vdash_\omega \mathtt{next}\omega \colon \blacktriangleright T} \qquad \frac{\Psi, \phi \colon \blacktriangleright T \vdash_\omega \omega \colon T}{\Psi \vdash_\omega \mathtt{fix}\ \phi.\omega \colon T}
$$

$$
\frac{\Psi, \phi \colon T_1 \vdash_\omega \omega \colon T_2}{\Psi \vdash_\omega \lambda\phi.\omega \colon T_1 \to T_2} \qquad \frac{\Psi \vdash_\omega \omega_1 \colon T_1 \quad \Psi \vdash_\omega \omega_2 \colon T_1 \to T_2}{\Psi \vdash_\omega \omega_2\omega_1 \colon T_2}
$$

Fig. 2. Typing rules of guarded symmetric pattern matching.

Remark 5.1 Components of morphisms in $(\mathbf{C}_\mathrm{e})^\infty$ are dagger epimorphisms; and isomorphic dagger epimorphism are dagger isomorphisms, the morphisms obtained by Theorem 3.12 then have dagger isomorphic components. Lemma 4.19 shows that they are even morphisms in $\mathbf{C}_\dagger^\infty$.

Terms admit only closed types. The interpretation of a closed type $\cdot \vdash A$ is a functor $1 \to (\mathbf{C}_\mathrm{e})^\infty$, therefore it is simply an object of $(\mathbf{C}_\mathrm{e})^\infty$ (and thus an object of $\mathbf{C}_\mathrm{e}^\infty$ and $\mathbf{C}_\dagger^\infty$ too). We abuse notation and

write $[\![A]\!]$ for the object of $\mathbf{C}_{\mathrm{e}}^{\infty}$ given by $[\![\cdot \vdash A]\!](*)$.

Example 5.2 The interpretation of the unit type `I` is the object of $\mathbf{C}_{\mathrm{e}}^{\infty}$, represented in $\mathbf{C}$ by: $I \xleftarrow{\mathrm{id}_I} I \xleftarrow{\mathrm{id}_I} I \xleftarrow{\mathrm{id}_I} I \longleftarrow \cdots$

Given a closed type A, the type of lists of type A is $\mu X.\mathtt{I} \oplus (A \otimes \blacktriangleright X)$. Its interpretation in $\mathbf{C}_{\mathrm{e}}^{\infty}$ is: $I \xleftarrow{\iota_1^\dagger} I \oplus [\![A]\!](1) \xleftarrow{r \circ \iota_1^\dagger} (I \oplus [\![A]\!](2)) \oplus [\![A]\!](2)^{\otimes 2} \xleftarrow{r \circ \iota_1^\dagger} \dots$

5.2 Semantics of Ground Terms

The primary terms of symmetric pattern matching are as expected. Typing judgements for terms have the form $\Delta \vdash t : A$, where $\Delta = x_1 : A_1, \dots, x_m : A_m$ is a context, and A is a closed type. The semantics of Δ is the tensor product of the semantics of all its components A_i.

$$\frac{}{\cdot \vdash * : I} \qquad \frac{\Delta_1 \vdash t_1 : A_1 \quad \Delta_2 \vdash t_2 : A_2}{\Delta_1, \Delta_2 \vdash t_1 \otimes t_2 : A_1 \otimes A_2} \qquad \frac{\Delta_1 \vdash t_1 : A \otimes B \quad \Delta_2, x : A, y : B \vdash t_2 : C}{\Delta_2, \Delta_1 \vdash \mathtt{let}\, x \otimes y = t_1 \ \mathtt{in}\ t_2 : C}$$

$$\frac{\Delta \vdash t : A_i}{\Delta \vdash \mathtt{inj}_i\, t : A_1 \oplus A_2} \qquad \frac{\vdash_\omega \omega : A \leftrightarrow B \quad \Delta \vdash t : A}{\Delta \vdash \omega\ t : B} \qquad \frac{\Delta \vdash t : A}{\Delta \vdash \mathtt{next}\, t : \blacktriangleright A}$$

$$\frac{\vdash_\omega \omega : \blacktriangleright(A \leftrightarrow B) \quad \Delta \vdash t : \blacktriangleright A}{\Delta \vdash \omega \circledcirc t : \blacktriangleright B} \qquad \frac{\Delta \vdash t : A[\mu X.A/X]}{\Delta \vdash \mathtt{fold}\, t : \mu X.A}$$

Example 5.3 If we have $\Delta \vdash t : [A]$ and $\Delta' \vdash h : A$, the list with head h and tail t is obtained in the syntax by $\mathtt{fold}\ \mathtt{inj}_2\ (h \otimes \mathtt{next}\, t)$.

The interpretation of a type judgement $\Delta \vdash t : A$ is a morphism $[\![\Delta \vdash t : A]\!] : [\![\Delta]\!] \to [\![A]\!]$ in $\mathbf{C}_{\mathrm{e}}^{\infty}$. The semantics of injections $\mathtt{inj}_i$ (resp. `next`, `fold`) is obtained by postcomposing with $\iota_i : [\![A_i]\!] \to [\![A_1]\!] \oplus [\![A_2]\!]$ (resp. $\nu^{\mathbf{C}}_{[\![A]\!]} : [\![A]\!] \to L^{\mathbf{C}}[\![A]\!]$, $\phi^{[\![X \vdash A]\!]}_{[\![\mu X.A]\!]} : [\![A[\mu X.A/X]]\!] \to [\![\mu X.A]\!]$). The interpretation for tensoring two terms is simply the tensor of the semantics of the terms. The semantics of the `let`, destructor of the tensor product, is given by $[\![t_2]\!] \circ (\mathrm{id}_{[\![\Delta_2]\!]} \otimes [\![t_1]\!])$.

Note that among the morphisms mentioned above, only $\nu^{\mathbf{C}}$ is not daggerable.

5.3 Semantics of First-Order Functions

The functions in symmetric pattern matching are called *isos* – short for 'isomorphisms', even though they are not necessarily isomorphic in some extensions of the language – and are formed as the *join* of several reversible partial functions.

$$\frac{\Delta_i \vdash t_i : A \quad \forall i \neq j, t_i \perp t_j \quad \Delta_i \vdash t'_i : B \quad \forall i \neq j, t'_i \perp t'_j}{\vdash_\omega \{t_1 \leftrightarrow t'_1 \ \mid \ \cdots \ \mid \ t_n \leftrightarrow t'_n\} : A \leftrightarrow B}$$

The most important premise in the typing rule above is pairwise orthogonality [46, Section 2.2], but the detail is not relevant in this paper. The type of iso abstractions takes its semantics as hom-objects $\mathbf{C}_\dagger^\infty([\![A]\!], [\![B]\!])$ – which is an object in $\mathbf{S}$ – since we want those functions to be reversible and their semantics to have a dagger. The interpretation of each subfunction forming an iso is given by: $[\![t_i \leftrightarrow t'_i]\!] = [\![\Delta \vdash t'_i : B]\!] \circ [\![\Delta \vdash t_i : A]\!]^\dagger$. As observed above in Remark 4.10, the latter is not necessarily well-defined. To overcome this issue, we introduce a symmetric binary relation – that we write $\bowtie$, say 'have same depth as' – in order to ensure that an iso is well-defined. First we introduce *next-free* contexts, with $\mathtt{next} \notin t$, as:

$$C[-] ::= - \mid \mathtt{inj}_i\, C[-] \mid C[-] \otimes t \mid t \otimes C[-] \mid \mathtt{fold}\, C[-] \mid \omega\ C[-]$$
$$\mid \mathtt{let}\, x \otimes y = C[-] \ \mathtt{in}\ t \mid \mathtt{let}\, x \otimes y = t \ \mathtt{in}\ C[-]$$

Definition 5.4 Given two terms t, t', we have $t \bowtie t'$ if it can be derived with the rules below. When $t \bowtie t'$ can be derived, we say that t and t' have the *same depth*. The relation $\bowtie$ is defined as the smallest symmetric relation on terms such that $* \bowtie *$ and:

$$\frac{t \bowtie t'}{C[t] \bowtie C'[t']} \quad \frac{t \bowtie t'}{\mathtt{next}\, t \bowtie \mathtt{next}\, t'} \quad \frac{t \bowtie t'}{\mathtt{next}\, \mathtt{inj}_i\, t \bowtie \mathtt{inj}_i\, \mathtt{next}\, t'} \quad \frac{t_1 \bowtie t_1' \quad t_2 \bowtie t_2'}{\mathtt{next}\, t_1 \otimes t_2 \bowtie \mathtt{next}\, t_1' \otimes \mathtt{next}\, t_2'}$$

with $C[-]$ and $C'[-]$ potentially different next-free contexts.

Lemma 5.5 *If $\Delta \vdash t : A$, $\Delta \vdash t' : B$ and $t \bowtie t'$, then $[\![t_i \leftrightarrow t_i']\!]$ is in $\mathbf{C}_\dagger^\infty([\![A]\!], [\![B]\!])$.*

We then assume that there is a *join* structure in $\mathbf{C}$, similar to the one in join inverse categories [2], or similar to the one used for contractions between Hilbert spaces [38, Lemma 3.37], or such as the summability for models of linear logic [22, Definition 4.5]. That is to say, if two parallel morphisms $f, g : X \to Y$ are so-called *compatible* – the definition does not matter here –, the join $f \vee g : X \to Y$ is also a morphism in $\mathbf{C}$. This join should be distributive with composition, *i.e.* we have $h \circ (f \vee g) = hf \vee hg$ and $(f \vee g) \circ h' = fh' \vee gh'$, and it should also be compatible with the dagger, *i.e.* we have $(f \vee g)^\dagger = f^\dagger \vee g^\dagger$. Given these conditions, this join structure generalises to $\mathbf{C}_\dagger^\infty$: say that two parallel morphisms $f, g : X \to Y$ in $\mathbf{C}_\dagger^\infty$ are compatible if their components are pointwise compatible. Naturality is ensured since the join distributes with composition. Condition (iii) is thus satisfied. The semantics of an iso abstraction is therefore defined in $\mathbf{C}_\dagger^\infty$ (thus fitting Point (iv)) as:

$$[\![\{t_1 \leftrightarrow t_1' \mid \cdots \mid t_n \leftrightarrow t_n'\} : A \leftrightarrow B]\!] = \bigvee_i [\![t_i \leftrightarrow t_i']\!] = \bigvee_i [\![\Delta \vdash t_i' : B]\!] \circ [\![\Delta \vdash t_i : A]\!]^\dagger .$$

Example 5.6 [Flip the first Boolean] Consider the following iso of type $[\mathtt{I} \oplus \mathtt{I}] \leftrightarrow [\mathtt{I} \oplus \mathtt{I}]$ that maps a list of Booleans to a list of Booleans where the first element is flipped:

$$\left\{ \begin{array}{ll} [\,] & \leftrightarrow [\,] \\ (\mathtt{inj}_1\, *) :: \mathtt{next}\, t & \leftrightarrow (\mathtt{inj}_2\, *) :: \mathtt{next}\, t \\ (\mathtt{inj}_2\, *) :: \mathtt{next}\, t & \leftrightarrow (\mathtt{inj}_1\, *) :: \mathtt{next}\, t \end{array} \right\}$$

5.4 Semantics of Higher-Order Types

Due to the enrichment of our categories (namely, $(\mathbf{C}_\mathrm{e})^\infty$, $\mathbf{C}_\mathrm{e}^\infty$ and $\mathbf{C}_\dagger^\infty$) in $\mathbf{S}$, we add a simply-typed guarded λ-calculus on top of the functions, in our syntax. Until now, the only function type was $A \leftrightarrow B$ given two closed term types A and B. We then allow for *functions of functions*, with the following type grammar: $T ::= A \leftrightarrow B \mid \blacktriangleright T \mid T_1 \to T_2$. These functions types take their semantics as objects in $\mathbf{S}$. The semantics of $A \leftrightarrow B$ is $\mathbf{C}_\dagger^\infty([\![A]\!], [\![B]\!])$. The semantics of $T_1 \to T_2$ is $[[\![T_1]\!] \to [\![T_2]\!]]$, the exponential object in $\mathbf{S}$.

5.5 Semantics of Higher-Order Terms and Functions

We add *function contexts* $\Psi = \phi_1 : T_1, \ldots, \phi_k : T_k$; now well-typed terms and functions can depend on a function context, with term judgements such as $\Psi; \Delta \vdash t : A$ and function judgements as $\Psi \vdash_\omega \omega : T$. An iso abstraction can be applied to a function, thus $\Psi; \Delta \vdash \omega\, t : B$ is well-formed whenever $\Psi \vdash_\omega \omega : A \leftrightarrow B$ and $\Psi; \Delta \vdash t : A$ are. The denotational semantics of term judgement is given as a morphism in $\mathbf{S}$: $[\![\Psi; \Delta \vdash t : A]\!] : [\![\Psi]\!] \to \mathbf{C}_\mathrm{e}^\infty([\![\Delta]\!], [\![A]\!])$. In addition, the grammar of functions is extended to a simply-typed λ-calculus.

$$\omega ::= \{t_1 \leftrightarrow t_1' \mid \cdots \mid t_n \leftrightarrow t_n'\} \mid \phi \mid \lambda\phi.\omega \mid \omega_2\omega_1 \mid \mathtt{fix}\, \phi.\omega \mid \mathtt{next}\omega \mid \omega_2 \circledcirc \omega_1$$

The semantics for a function judgement $\Psi \vdash_\omega \omega \colon T$ is a morphism in $\mathbf{S}$ with type $[\![\Psi]\!] \to [\![T]\!]$. Therefore the semantics of the new terms are the usual ones for a simply-typed λ-calculus in a cartesian closed category, as pointed out in (v).

The structure of $\mathbf{S}$ allows for the addition of several other kinds of functions terms, inherited from the guarded lambda calculus: a delay operation $\texttt{next}\,\omega$, whose semantics is $\nu^{\mathbf{Set}}$ and a composition of delayed operations $\omega_2 \circledcirc \omega_1$. A delayed iso $\omega \colon \blacktriangleright(A \leftrightarrow B)$ can also be applied to a delayed term $t \colon \blacktriangleright A$ to obtain $\omega \circledcirc t \colon \blacktriangleright B$.

As shown in §2.2, there is a fixed point operator linked to the guarded structure in $\mathbf{S}$. We then add to the syntax of functions a fixed point operator $\texttt{fix}$, as desired in Point (vi):

$$\frac{\Psi, \phi \colon \blacktriangleright T \vdash_\omega \omega \colon T}{\Psi \vdash_\omega \texttt{fix}\ \phi.\omega \colon T} \tag{4}$$

The morphism $[\![\Psi, \phi \colon \blacktriangleright T \vdash_\omega \omega \colon T]\!]$ is contractive on its last variable, and therefore admits a guarded fixed point [5, Theorem 2.4], used as the semantics for the term $\Psi \vdash_\omega \texttt{fix}\ \phi.\omega \colon T$.

Example 5.7 Given A and B two types, we have now a proper semantics for the function $\texttt{map}$ which applies a function $A \leftrightarrow B$ to all the element of a list $[A]$.

$$\texttt{map} = \texttt{fix}\ \phi^{\blacktriangleright((A\leftrightarrow B)\to([A]\leftrightarrow[B]))}.\lambda\psi^{A\leftrightarrow B}.\left\{\begin{array}{ll}[\,] & \leftrightarrow [\,] \\ h :: t & \leftrightarrow (\psi\ h) :: ((\phi \circledcirc \texttt{next}\,\psi) \circledcirc t)\end{array}\right\}$$

5.6 The Pure Quantum Case

As mentioned above, the semantics of symmetric pattern matching [46] with inductive types and higher-order quantum operations is an open question. There are multiple factors that make this question hard, *e.g.* categories of Hilbert spaces are unlikely to interpret recursion (see [38, Section 5.2]). Guarded recursion helps bypass those issues.

Example 5.8 [Quantum control] In pure quantum computing, the type of quantum bits (or *qubits*) is given by $\texttt{qubit} \stackrel{\text{def}}{=} \texttt{I} \oplus \texttt{I}$. The term $\texttt{inj}_1\ * \colon \texttt{qubit}$ represents the qubit $|0\rangle$, and $\texttt{inj}_2\ *$, the qubit $|1\rangle$. The high-order formalism of symmetric pattern matching allows for a general control operation $\texttt{qctrl} \colon (A \leftrightarrow B) \to (A \leftrightarrow B) \to (\texttt{qubit} \otimes A \leftrightarrow \texttt{qubit} \otimes B)$, that we refer to as *quantum control* or *quantum if*:

$$\texttt{qctrl} = \lambda\phi^{A\leftrightarrow B}.\lambda\psi^{A\leftrightarrow B}.\left\{\begin{array}{l}|0\rangle \otimes y \leftrightarrow |0\rangle \otimes (\phi\ y) \\ |1\rangle \otimes y \leftrightarrow |1\rangle \otimes (\psi\ y)\end{array}\right\}$$

It is a *quantum if*, because it applies either ϕ or ψ to the second qubit depending on the value of the first qubit, without measuring it.

Example 5.9 [General QFT] The *Quantum Fourier Transform* is a subroutine that plays a central role in quantum algorithms. In the current literature, a quantum algorithm is often stated in the form of quantum circuits on a fixed number of qubits. We can here abstract away from this low-level description, and offer more appealing view for programming languages, by defining a quantum Fourier transform subroutine for lists of qubits.

First, given an iso $\omega \colon A \leftrightarrow A$, we define $\omega \circ \omega$ as the iso abstraction $\{x \leftrightarrow \omega\ (\omega\ x)\} \colon A \leftrightarrow A$. We write $\texttt{2}$ for the type of qubits $\texttt{I} \oplus \texttt{I}$ for space reasons. We can then write the iso that applies gradual rotations to a list of qubits; here, it eventually applies ψ^k to the k^{th} element of the list (with $T = \blacktriangleright(\texttt{2} \leftrightarrow \texttt{2}) \to (\texttt{2} \leftrightarrow \texttt{2}) \to ([\texttt{2}] \leftrightarrow [\texttt{2}])$):

$$\texttt{Rot} = \texttt{fix}\ \phi^T.\lambda\psi^{\texttt{2}\leftrightarrow\texttt{2}}.\lambda\psi'^{\texttt{2}\leftrightarrow\texttt{2}}.\left\{\begin{array}{ll}[\,] & \leftrightarrow [\,] \\ h :: t & \leftrightarrow (\psi'\ h) :: (\phi \circledcirc \texttt{next}\,\psi \circledcirc \texttt{next}\,(\psi \circ \psi')) \circledcirc t)\end{array}\right\}$$

then, given $\texttt{had}\colon 2 \leftrightarrow 2$ the Hadamard gate, we have (with $T' = \blacktriangleright((2 \leftrightarrow 2) \to ([2] \leftrightarrow [2]))$):

$$\texttt{QFT} = \texttt{fix}\ \phi^{\blacktriangleright((2\leftrightarrow 2)\to([2]\leftrightarrow[2]))}.\lambda\psi^{2\leftrightarrow 2}.\left\{\begin{array}{rl} [\,] & \leftrightarrow [\,] \\ h :: \texttt{next}\, t & \leftrightarrow \texttt{let}\, h' \otimes t' = \texttt{qctrl}(\texttt{Rot}(\psi))\ ((\texttt{had}\ h) \otimes t)\ \texttt{in} \\ & \quad h' :: ((\phi \circledcirc \texttt{next}\, \psi) \circledcirc \texttt{next}\, t') \end{array}\right\}$$

This offers a higher-order perspective on reversible quantum programming.

5.7 What about streams?

The story of reversible guarded recursion is very different to the classical one. For example, streams are usually computed as a fixed point of the functor $X \otimes T$, obtained as the limit of the following diagram, assuming T is a terminal object.

$$T \xleftarrow{!} X \otimes T \xleftarrow{X\otimes !} X \otimes X \otimes T \longleftarrow \cdots$$

Lemma 5.10 *Given a dagger category* $\mathbf{C}$*, if* T *is a terminal object in* $\mathbf{C}$*, then it is also initial.*

Therefore, in a dagger rig category, a terminal object is also a zero object, and the diagram above falls down to a cosequence of zero objects. whose limit is also the zero object. This does not necessarily imply that streams cannot be manipulated reversibly; but the semantics of streams certainly cannot be computed in a usual way in a dagger category. Working with reversible streams then requires a new theory.

Acknowledgments

I would like to thank Benoît Valiron and Vladimir Zamdzhiev for their support and helpful comments during my thesis; and thanks to Robert Booth, Titouan Carette, Kostia Chardonnet, Pierre Clairambault, Jonas Frei, Laurent Regnier, and Morgan Rogers for our chats about this topic. I am also grateful to Chris Heunen, Robin Kaarsgaard and Kim Worrall, and more generally the Quantum Programming Group in the University of Edinburgh for their support and feedback.

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